NONCOMMUTATIVE RIGIDITY OF HIGHER RANK LATTICES

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Abstract. These are the notes of a lecture series given at the Henri Poincaré Institute in June 2024 during the trimester *Group actions and rigidity: around the Zimmer program*. We survey recent results regarding dynamics of positive definite functions and character rigidity of higher rank lattices. We discuss the notion of noncommutative boundary structure and we give the proof of the noncommutative Nevo–Zimmer structure theorem for ergodic actions of higher rank lattices on von Neumann algebras due to Boutonnet–Houdayer. We present several applications to ergodic theory, topological dynamics, unitary representation theory and operator algebras. We also present a noncommutative analogue of Margulis' factor theorem for higher rank lattices and we explain its relevance towards Connes' celebrated rigidity conjecture.

1. Lecture 1: Dynamics of positive definite functions

In the first lecture, we give a brief introduction to positive definite functions, unitary representations and operator algebras. We state our main results regarding dynamics on the space of positive definite functions, character rigidity and structure of group C*-algebras for higher rank lattices [BH19, BBHP20]. We also present a noncommutative analogue of Margulis' factor theorem for higher rank lattices and we explain its relevance towards Connes' celebrated rigidity conjecture [Ho21, BH22].

1.1. **Introduction and motivation.**
Throughout these lectures, we use the following terminology regarding *higher rank lattices*.

Terminology. Let G be a semisimple connected real Lie group with finite center, no nontrivial compact factor and real rank $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$. Let $\Gamma < G$ be an *irreducible lattice*, meaning that Γ is a discrete subgroup of G with finite covolume such that $N \cdot \Gamma$ is a dense subgroup of G for every noncentral closed normal subgroup $N \lhd G$. In what follows, if all the above conditions are satisfied, then we simply say that $\Gamma < G$ is a *higher rank lattice*.

The following examples of higher rank lattices are particular cases of general results due to Borel–Harish-Chandra [BHC61].

The author is supported by Institut Universitaire de France.

Examples. For every $d \geq 2$, the special linear group $\mathrm{SL}_d(\mathbb{R})$ is a simple connected real Lie group with finite center $\mathscr{Z}(\mathrm{SL}_d(\mathbb{R})) = \{\pm 1_d\}$ and real rank $\mathrm{rk}_{\mathbb{R}}(\mathrm{SL}_d(\mathbb{R})) = d - 1$.

- For every $d \geq 3$, $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is a higher rank lattice.
- For every $d \geq 2$ and every square-free integer $q \in \mathbb{N} \setminus \{0, 1\}$,
$$\Gamma = \{(g, g^\sigma) \mid g \in \mathrm{SL}_d(\mathbb{Z}[\sqrt{q}])\} < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R}) = G$$
is a higher rank lattice, where σ is the order 2 automorphism of $\mathbb{Q}(\sqrt{q})$.

Among the major achievements in the study of higher rank lattices, there are two groundbreaking results by Margulis that are relevant to these lectures.

Firstly, Margulis' *normal subgroup theorem* states that for any higher rank lattice $\Gamma < G$, any normal subgroup $N \lhd \Gamma$ is either finite and contained in $\mathscr{Z}(\Gamma)$ or N has finite index in Γ (see [Ma91, Theorem IV.4.9]). Margulis' remarkable strategy to prove the normal subgroup theorem consists of two parts: the *amenability half* and the *property* (T) *half*. Indeed, assuming that $N \lhd \Gamma$ is a noncentral normal subgroup, to prove that the quotient group Γ/N is finite, Margulis showed that Γ/N is amenable and has property (T). The proof of the amenability half relies on Margulis' *factor theorem* which states that any measurable Γ-factor of the homogeneous space G/P, where $P < G$ is a minimal parabolic subgroup, is measurably Γ-isomorphic to G/Q, where $P < Q < G$ is an intermediate parabolic subgroup (see [Ma91, Theorem IV.2.11]).

Secondly, Margulis' *superrigidity theorem* states that for any higher rank lattice $\Gamma < G$ and any simple algebraic k-group $\mathbf{H}$, where k is a local field, any group homomorphism $\pi : \Gamma \to \mathbf{H}(k)$ such that $\pi(\Gamma) < \mathbf{H}(k)$ is unbounded and Zariski dense extends uniquely to a continuous group homomorphism $\pi : G \to \mathbf{H}(k)$ (see [Ma91, Chapter VII]). Margulis' superrigidity theorem has two fundamental applications: Mostow–Margulis' *strong rigidity theorem* and Margulis' *arithmeticity theorem* (see [Ma91, Chapter VIII]).

1.2. **Operator algebras and unitary representation theory.** A C*-*algebra* A is a Banach $*$-algebra endowed with a complete norm $\|\cdot\|$ that satisfies the following C*-identity
$$\forall a \in A, \quad \|a^*a\| = \|a\|^2 = \|aa^*\|.$$
Any C*-algebra A admits a faithful isometric $*$-representation on a Hilbert space $\pi : A \to \mathrm{B}(\mathscr{H})$. After identifying A with $\pi(A)$, we may regard $A \subset \mathrm{B}(\mathscr{H})$ as a concrete C*-algebra. Unless stated otherwise, all C*-algebras and all linear mappings between C*-algebras are always assumed to be unital.

We denote by $\mathfrak{S}(A)$ the *state space* of A. Then $\mathfrak{S}(A) \subset \mathrm{Ball}(A^*)$ is a weak-$*$ compact convex subset. We say that an action $\sigma : H \curvearrowright A$ is *continuous* if the action map $H \times A \to A : (g, a) \mapsto \sigma_g(a)$ is continuous. We then simply say that A is a H-C*-algebra. The continuous action $H \curvearrowright A$ induces a weak-$*$ continuous affine action $H \curvearrowright \mathfrak{S}(A)$.

Examples. We will consider the following examples of C*-algebras.

(i) For any compact metrizable space X, the space $\mathrm{C}(X)$ of all continuous functions on X endowed with the uniform norm $\|\cdot\|_\infty$ is a commutative C*-algebra. Any commutative C*-algebra arises this way. We identify the set $\mathrm{Prob}(X)$ of Borel probability measures on X with the state space $\mathfrak{S}(\mathrm{C}(X))$ via the continuous mapping $\mathrm{Prob}(X) \to \mathfrak{S}(\mathrm{C}(X)) : \nu \mapsto \int_X \cdot \, \mathrm{d}\nu$. Any continuous action by homeomorphisms $H \curvearrowright X$ naturally gives rise to a continuous action $H \curvearrowright \mathrm{C}(X)$ in the above sense.

(ii) For any countable discrete group Λ and any unitary representation $\pi : \Lambda \to \mathscr{U}(\mathscr{H}_\pi)$, define the C*-algebra
$$\mathrm{C}^*_\pi(\Lambda) = \mathrm{C}^*(\{\pi(\gamma) \mid \gamma \in \Lambda\}) \subset \mathrm{B}(\mathscr{H}_\pi)$$
and consider the conjugation action $\mathrm{Ad}(\pi) : \Lambda \curvearrowright \mathrm{C}^*_\pi(\Lambda)$. If $\pi = \lambda : \Lambda \to \mathscr{U}(\ell^2(\Lambda))$ is the left regular representation, then $\mathrm{C}^*_\lambda(\Lambda)$ is the *reduced* group C*-algebra. In that case, the state $\tau_\Lambda : \mathrm{C}^*_\lambda(\Lambda) \to \mathbb{C} : a \mapsto \langle a\delta_e, \delta_e\rangle$ is a faithful trace.

We also consider the *full* C*-algebra $\mathrm{C}^*(\Lambda)$ that is the C*-completion of the group algebra $\mathbb{C}[\Lambda]$ with respect to the uniform norm $\|\cdot\|_u$ defined by
$$\forall a \in \mathbb{C}[\Lambda], \quad \|a\|_u = \sup\left\{\|\pi(a)\|_{\mathrm{B}(\mathscr{H}_\pi)} \mid \pi : \Lambda \to \mathscr{U}(\mathscr{H}_\pi) \text{ cyclic}\right\}.$$
Then for every unitary representation $\pi : \Lambda \to \mathscr{U}(\mathscr{H}_\pi)$, we may extend π to a surjective unital $*$-homomorphism $\pi : \mathrm{C}^*(\Lambda) \to \mathrm{C}^*_\pi(\Lambda)$ and we have $\mathrm{C}^*_\pi(\Lambda) \cong \mathrm{C}^*(\Lambda)/\ker(\pi)$. In particular, when $\pi = 1_\Lambda$ is the trivial representation, we obtain a unital $*$-homomorphism $\chi : \mathrm{C}^*(\Lambda) \to \mathbb{C}$ such that $\chi(\gamma) = 1$ for every $\gamma \in \Lambda$. The countable discrete group Λ is amenable if and only if the surjective unital $*$-homomorphism $\mathrm{C}^*(\Lambda) \to \mathrm{C}^*_\lambda(\Lambda)$ is an isomorphism.

For every $i \in \{1, 2\}$, let $\pi_i : \Lambda \to \mathscr{U}(\mathscr{H}_i)$ be a unitary representaton that we extend to a surjective unital $*$-homomorphism $\pi_i : \mathrm{C}^*(\Lambda) \to \mathrm{C}^*_{\pi_i}(\Lambda)$. We say that π_2 is *weakly contained* in π_1 and write $\pi_2 \prec \pi_1$ if one of the following equivalent assertions hold:

(i) $\ker(\pi_1) \subset \ker(\pi_2)$.
(ii) For every $a \in \mathrm{C}^*(\Lambda)$, we have $\|\pi_2(a)\|_{\mathrm{B}(\mathscr{H}_2)} \leq \|\pi_1(a)\|_{\mathrm{B}(\mathscr{H}_1)}$.

In that case, there is a unique surjective unital $*$-homomorphism $\Theta_{\pi_1,\pi_2} : \mathrm{C}^*_{\pi_1}(\Lambda) \to \mathrm{C}^*_{\pi_2}(\Lambda)$ such that $\pi_2 = \Theta_{\pi_1,\pi_2} \circ \pi_1$. We say that π_1 and π_2 are *weakly equivalent* and write $\pi_1 \sim \pi_2$ if $\pi_2 \prec \pi_1$ and $\pi_1 \prec \pi_2$. Then we have $\pi_1 \sim \pi_2$ if and only if $\ker(\pi_1) = \ker(\pi_2)$.

A *von Neumann algebra* (or W*-*algebra*) M is a unital C*-algebra which admits a faithful unital $*$-representation $\pi : M \to \mathrm{B}(\mathscr{H})$ such that $\pi(M) \subset \mathrm{B}(\mathscr{H})$ is closed with respect to the weak (equivalently strong) operator topology. After identifying M with $\pi(M)$, we may regard $M \subset \mathrm{B}(\mathscr{H})$ as a concrete von Neumann algebra. By von Neumann's bicommutant theorem, a unital $*$-subalgebra $M \subset \mathrm{B}(\mathscr{H})$ is a von Neumann algebra if and only if M is equal to its own bicommutant $M'' = (M')'$, that is, $M = M''$. There is a unique Banach space predual M_* such that $M = (M_*)^*$. The ultraweak topology on M coincides with the weak-$*$ topology arising from the identification $M = (M_*)^*$. A linear mapping between von Neumann algebras is *normal* if it is continuous with respect to the ultraweak topology. We say that an action $\sigma : H \curvearrowright M$ is *continuous* if the corresponding action map $H \times M_* \to M_* : (g, \varphi) \mapsto \varphi \circ \sigma_g^{-1}$ is continuous (see e.g. [Ta03a, Proposition X.1.2]). We then simply say that M is a H-von Neumann algebra. We will consider the subset $A \subset M$ of all H-*continuous elements*

$$A = \left\{ x \in M \mid \lim_{h \to e} \|\sigma_h(x) - x\|_\infty = 0 \right\}.$$

Then $A \subset M$ is a H-invariant ultraweakly dense unital C^*-subalgebra for which the action $H \curvearrowright A$ is $\| \cdot \|_\infty$-continuous (see e.g. [Ta03b, Proposition XIII.1.2]). We say that $A \subset M$ is the H-*continuous model of* M. The action $H \curvearrowright M$ is *ergodic* if the fixed point von Neumann subalgebra $M^H = \{x \in M \mid \forall g \in H, \sigma_h(x) = x\}$ is trivial.

Examples. We will consider the following examples of von Neumann algebras.

(i) For any standard probability space (X, ν), the space $\mathrm{L}^\infty(X, \nu)$ of all ν-equivalence classes of (essentially) bounded measurable functions endowed with the (essential) uniform norm $\| \cdot \|_\infty$ is a commutative von Neumann algebra and we have $\mathrm{L}^\infty(X, \nu) \cong \mathrm{L}^1(X, \nu)^*$. Any commutative von Neumann algebra arises this way. Any nonsingular action $H \curvearrowright (X, \nu)$ naturally gives rise to a continuous action $H \curvearrowright \mathrm{L}^\infty(X, \nu)$ in the above sense. When no confusion is possible, we simply write $\mathrm{L}^\infty(X) = \mathrm{L}^\infty(X, \nu)$. For any von Neumann algebra M with separable predual, we identify $\mathrm{L}^\infty(X) \mathbin{\overline{\otimes}} M \cong \mathrm{L}^\infty(X, M)$.

Let H be a lcsc group and $Q < H$ a closed subgroup. Consider the continuous action $H \curvearrowright H/Q$ together with the unique H-invariant measure class on H/Q. For any von Neumann algebra M with separable predual, the H-continuous model of $\mathrm{L}^\infty(H/Q, M)$ is contained in the unital C^*-algebra $\mathrm{C}_b(H/Q, M)$.

(ii) For any countable discrete group Λ and any unitary representation $\pi : \Lambda \to \mathscr{U}(\mathscr{H}_\pi)$, define the von Neumann algebra

$$\mathrm{W}^*_\pi(\Lambda) = \{\pi(\gamma) \mid \gamma \in \Lambda\}'' \subset \mathrm{B}(\mathscr{H}_\pi)$$

and consider the conjugation action $\mathrm{Ad}(\pi) : \Lambda \curvearrowright \mathrm{W}^*_\pi(\Lambda)$. If $\pi = \lambda : \Lambda \to \mathscr{U}(\ell^2(\Lambda))$ is the left regular representation, then $\mathrm{W}^*_\lambda(\Lambda)$ is the group von Neumann algebra and is usually denoted by $\mathrm{L}(\Lambda) = \mathrm{W}^*_\lambda(\Lambda)$. Moreover, the state $\tau_\Lambda : \mathrm{L}(\Lambda) \to \mathbb{C} : a \mapsto \langle a\delta_e, \delta_e\rangle$ is a faithful normal trace.

(iii) For any countable discrete group Λ and any nonsingular action $\Lambda \curvearrowright (X, \nu)$ on a standard probability space, define the *group measure* *space von Neumann algebra*

$$\mathrm{L}(\Lambda \curvearrowright X) = \{f \otimes 1, \pi(\gamma) \mid f \in \mathrm{L}^\infty(X), \gamma \in \Lambda\}'' \subset \mathrm{B}(\mathrm{L}^2(X,\nu) \otimes \ell^2(\Lambda))$$

where $\pi : \Lambda \to \mathscr{U}(\mathrm{L}^2(X,\nu) \otimes \ell^2(\Lambda))$ is the unitary representation defined by

$$\forall \xi \in \mathrm{L}^2(X,\nu), \forall \gamma, h \in \Lambda, \quad \pi(\gamma)(\xi \otimes \delta_h) = \sqrt{\frac{\mathrm{d}(\nu \circ \gamma^{-1})}{\mathrm{d}\nu}}\, \xi \circ \gamma^{-1} \otimes \delta_{\gamma h}.$$

When (X, ν) is a singleton, the von Neumann algebra $\mathrm{L}(\Lambda \curvearrowright X)$ coincides with the *group von Neumann algebra* $\mathrm{L}(\Lambda)$.

If the group Λ is icc (i.e. with infinite conjugacy classes), then the von Neumann algebra $\mathrm{L}(\Lambda)$ is a type II_1 factor.

If the action $\Lambda \curvearrowright (X, \nu)$ is (essentially) free and ergodic, then the von Neumann algebra $\mathrm{L}(\Lambda \curvearrowright X)$ is a factor whose type coincides with the type of the action (see e.g. [Ta03b, Theorem XIII.1.7]).

A von Neumann algebra $M \subset \mathrm{B}(\mathscr{H})$ is *amenable* if there exists a norm one projection $\mathrm{E} : \mathrm{B}(\mathscr{H}) \to M$. For any countable discrete group Λ, the group von Neumann algebra $\mathrm{L}(\Lambda)$ is amenable if and only if Λ is amenable. By Connes' fundamental result [Co75], M is amenable if and only if M is *approximately finite dimensional*, that is, there exists an increasing net of finite dimensional subalgebras $M_i \subset M$ such that $\bigvee_{i \in I} M_i = M$.

Following [Be89], we say that a unitary representation $\pi : \Lambda \to \mathscr{U}(\mathscr{H}_\pi)$ is *amenable* if the trivial representation 1_Λ is weakly contained in $\pi \otimes \overline{\pi}$ or equivalently if there exists an $\mathrm{Ad}(\pi)$-invariant state $\Psi \in \mathrm{B}(\mathscr{H}_\pi)^*$. If $\mathrm{W}^*_\pi(\Lambda)$ is amenable and carries a (normal) trace, then π is amenable. If π contains a finite dimensional subrepresentation, then π is amenable. If Λ has property (T), then conversely any amenable representation $\pi : \Lambda \to \mathscr{U}(\mathscr{H}_\pi)$ contains a finite dimensional subrepresentation.

1.3. **Dynamics of positive definite functions and applications.** Let Λ be a countable discrete group. We say that $\varphi : \Lambda \to \mathbb{C}$ is a *positive definite function* if for all $n \geq 1$ and all $\gamma_1, \dots, \gamma_n \in \Lambda$, the matrix $[\varphi(\gamma_i^{-1}\gamma_j)]_{i,j} \in \mathrm{M}_n(\mathbb{C})$ is positive semidefinite. We denote by $\mathscr{P}(\Lambda)$ the space of all normalized positive definite functions $\varphi : \Lambda \to \mathbb{C}$ so that $\varphi(e) = 1$. Then $\mathscr{P}(\Lambda) \subset \ell^\infty(\Lambda)$ is a weak-$*$ compact convex subset. One may view $\mathscr{P}(\Lambda)$ as the state space $\mathfrak{S}(\mathrm{C}^*(\Lambda))$ of the full C^*-algebra $\mathrm{C}^*(\Lambda)$. Thanks to the Gelfand–Naimark–Segal (GNS) construction, to any normalized positive definite function $\varphi \in \mathscr{P}(\Lambda)$ corresponds a triple $(\pi_\varphi, \mathscr{H}_\varphi, \xi_\varphi)$, where $\pi_\varphi : \Lambda \to \mathscr{U}(\mathscr{H}_\varphi)$ is a unitary representation and $\xi_\varphi \in \mathscr{H}_\varphi$ is a unit vector such that the linear span of $\pi_\varphi(\Lambda)\xi_\varphi$ is dense in $\mathscr{H}_\varphi$ and $\varphi(\gamma) = \langle \pi_\varphi(\gamma)\xi_\varphi, \xi_\varphi \rangle$ for every $\gamma \in \Lambda$. Consider the affine conjugation action $\Lambda \curvearrowright \mathscr{P}(\Lambda)$ defined by

$$\forall \gamma, g \in \Lambda, \forall \varphi \in \mathscr{P}(\Lambda), \quad (\gamma\varphi)(g) = \varphi(\gamma^{-1} g \gamma).$$

A fixed point $\varphi \in \mathscr{P}(\Lambda)$ for the conjugation action is called a *trace*. Denote by $\mathscr{T}(\Lambda) \subset \mathscr{P}(\Lambda)$ the weak-$*$ compact convex subset of all traces. One may view $\mathscr{T}(\Lambda)$ as the trace space $\mathscr{T}(\mathrm{C}^*(\Lambda))$ of the full C*-algebra $\mathrm{C}^*(\Lambda)$. It is known that $\mathscr{T}(\Lambda)$ is a Choquet simplex. For every $\tau \in \mathscr{T}(\Lambda)$, the *tracial* von Neumann algebra $\pi_\tau(\Lambda)''$ carries a faithful normal trace that is still denoted by τ. An extreme point $\tau \in \mathscr{T}(\Lambda)$ in the space of traces is called a *character*. Denote by $\mathscr{C}(\Lambda) \subset \mathscr{T}(\Lambda)$ the subset of all characters. For every $\tau \in \mathscr{T}(\Lambda)$, one has $\tau \in \mathscr{C}(\Lambda)$ if and only if the von Neumann algebra $\pi_\tau(\Lambda)''$ is a factor. Any icc group Λ admits at least two characters: the *trivial* character 1_Λ and the *regular* character δ_e. The GNS representation of the regular character δ_e coincides with the left regular representation $\lambda : \Lambda \to \mathscr{U}(\ell^2(\Lambda))$. For any irreducible finite dimensional unitary representation $\pi : \Lambda \to \mathscr{U}(n)$, the function $\varphi = \mathrm{tr}_n \circ \pi \in \mathscr{C}(\Lambda)$ is called a *compact* character.

Interesting examples of positive definite functions and traces come from group theory and ergodic theory. Consider the space $\mathrm{Sub}(\Lambda)$ of all subgroups of Λ endowed with the Chabauty topology and the conjugation action. Then $\mathrm{Sub}(\Lambda)$ is a compact metrizable space and we may regard $\mathrm{Sub}(\Lambda) \subset \{0,1\}^\Lambda$ as a closed Λ-invariant subset by identifying a subgroup $H < \Lambda$ with its characteristic function $\mathbf{1}_H \in \{0,1\}^\Lambda$. The canonical map

$$\theta : \mathrm{Sub}(\Lambda) \to \mathscr{P}(\Lambda) : H \mapsto \mathbf{1}_H$$

is continuous and Λ-equivariant. Observe that for every $H \in \mathrm{Sub}(\Lambda)$, we may consider the quasi-regular representation $\lambda_{\Lambda/H} : \Lambda \to \mathscr{U}(\ell^2(\Lambda/H))$ and $(\lambda_{\Lambda/H}, \ell^2(\Lambda/H), \delta_H)$ is the GNS triple of $\mathbf{1}_H \in \mathscr{P}(\Lambda)$. Then for every $H \in \mathrm{Sub}(\Lambda)$, $H \lhd \Lambda$ is a normal subgroup if and only if $\mathbf{1}_H \in \mathscr{T}(\Lambda)$ is a trace.

We consider the composition of the Λ-equivariant pushforward map

$$\mathrm{Prob}(\mathrm{Sub}(\Lambda)) \to \mathrm{Prob}(\mathscr{P}(\Lambda)) : \nu \mapsto \theta_*\nu$$

together with the Λ-equivariant barycenter map

$$\mathrm{Prob}(\mathscr{P}(\Lambda)) \to \mathscr{P}(\Lambda) : \psi \mapsto \mathrm{Bar}(\psi)$$

and we obtain the Λ-equivariant affine continuous map

$$\beta : \mathrm{Prob}(\mathrm{Sub}(\Lambda)) \to \mathscr{P}(\Lambda) : \nu \mapsto \mathrm{Bar}(\theta_*\nu).$$

Following [AGV12], an *invariant random subgroup* of Λ (or IRS for short) is a Λ-invariant Borel probability measure $\nu \in \mathrm{Prob}_\Lambda(\mathrm{Sub}(\Lambda))$. Note that if $\nu \in \mathrm{Prob}_\Lambda(\mathrm{Sub}(\Lambda))$ is an IRS, then $\beta(\nu) \in \mathscr{T}(\Lambda)$ is a trace. For any probability measure preserving (pmp) action $\Lambda \curvearrowright (X, \eta)$ on a standard probability space, the stabilizer map $\mathrm{Stab} : X \to \mathrm{Sub}(\Lambda) : x \mapsto \mathrm{Stab}_\Lambda(x)$ is Λ-equivariant and measurable (see [AM66]). Then it follows that $\nu = \mathrm{Stab}_*\eta \in \mathrm{Prob}_\Lambda(\mathrm{Sub}(\Lambda))$ is an IRS and we have $\beta(\nu)(\gamma) = \eta(\{x \in X \mid \gamma x = x\})$ for every $\gamma \in \Lambda$. The action $\Lambda \curvearrowright (X, \eta)$ is (essentially) free if and only if $\beta(\nu) = \delta_e$.

Following [GW14], a *uniformly recurrent subgroup* of Λ (or URS for short) is a nonempty Λ-invariant closed minimal subset $X \subset \mathrm{Sub}(\Lambda)$.

We now discuss our main results regarding the dynamics of positive definite functions and its applications to noncommutative rigidity of higher rank lattices (see [BH19, BBHP20] and the recent ICM survey [Ho21]). Our first main result deals with the *existence of traces*. It is a fixed point theorem for the affine action $\Gamma \curvearrowright \mathscr{P}(\Gamma)$ of higher rank lattices on the space of positive definite functions.

Theorem A ([BH19, BBHP20])**.** *Let $\Gamma < G$ be a higher rank lattice. Then any nonempty Γ-invariant weak-$*$ compact convex subset $\mathscr{C} \subset \mathscr{P}(\Gamma)$ contains a trace.*

Our second main result deals with the *classification of traces* of higher rank lattices. Bekka [Be06] obtained the first character rigidity results in the case $\Gamma = \mathrm{SL}_d(\mathbb{Z})$ for $d \geq 3$. More recently, using a different approach based on Margulis' strategy discussed above, Peterson [Pe14] obtained character rigidity results for arbitrary higher rank lattices. The operator algebraic framework we developed in [BH19, BBHP20] enables us to obtain a new and more conceptual proof of Peterson's character rigidity results [Pe14].

Theorem B (Peterson, [Pe14])**.** *Let $\Gamma < G$ be a higher rank lattice. Then any trace $\tau \in \mathscr{T}(\Gamma)$ is either supported on $\mathscr{Z}(\Gamma)$ or its* GNS *representation π_τ is amenable.*

In case G has a simple factor with property (T)*, any trace $\tau \in \mathscr{T}(\Gamma)$ is either supported on $\mathscr{Z}(\Gamma)$ or its* GNS *representation π_τ contains a finite dimensional subrepresentation.*

Theorem B generalizes Margulis' normal subgroup theorem [Ma91], Stuck–Zimmer's stabilizer rigidity theorem [SZ92] and solves a conjecture formulated by Connes in the early eighties (see [Jo00]). Moreover, by combining Theorems A and B, we obtain a positive solution to a problem raised by Glasner–Weiss on finiteness of URS of higher rank lattices [GW14].

Corollary C. *Let $\Gamma < G$ be a higher rank lattice and assume that G has a simple factor with property* (T)*. The following assertions hold:*

(i) *Let $N \lhd \Gamma$ be a normal subgroup. Then either $N \subset \mathscr{Z}(\Gamma)$ and N is finite or $[\Gamma : N] < +\infty$.*
(ii) *Let $\Gamma \curvearrowright (X, \eta)$ be an ergodic pmp action. Then either (X, η) is finite or $\mathrm{Stab}_\Gamma(x) \subset \mathscr{Z}(\Gamma)$ for η-almost every $x \in X$. In particular, any ergodic* IRS *has finite support.*
(iii) *Let $X \subset \mathrm{Sub}(\Gamma)$ be a* URS*. Then X is finite.*

Proof. (i) Let $N \lhd \Gamma$ be a normal subgroup and consider the trace $\tau = \mathbf{1}_N \in \mathscr{T}(\Gamma)$. Applying Theorem B, either τ is supported on $\mathscr{Z}(\Gamma)$ and so $N \subset \mathscr{Z}(\Gamma)$ is finite or the GNS representation $\lambda_{\Gamma/N}$ contains a finite dimensional subrepresentation. In the latter case, $\lambda_{\Gamma/N}$ is finite dimensional and so $[\Gamma : N] < +\infty$.

(ii) Let $\Gamma \curvearrowright (X, \eta)$ be an ergodic pmp action. Assume that (X, η) is not finite. By ergodicity, (X, η) is diffuse and η-almost every orbit is infinite. Consider the trace $\tau = \beta(\nu) \in \mathscr{T}(\Gamma)$ where $\nu = \mathrm{Stab}_* \eta \in \mathrm{Prob}_\Gamma(\mathrm{Sub}(\Gamma))$.

Define the ergodic pmp equivalence relation

$$\mathscr{R} = \{(\gamma x, x) \mid \gamma \in \Gamma, x \in X\} \subset X \times X.$$

Define the σ-finite Borel measure $m : \mathscr{B}(\mathscr{R}) \to \mathbb{R}_+$ by the formula

$$\forall \mathscr{W} \in \mathscr{B}(\mathscr{R}), \quad m(\mathscr{W}) = \int_X |\mathscr{W} \cap \{(\gamma x, x) \mid \gamma \in \Gamma\}| \, \mathrm{d}\nu(x).$$

Consider the infinite measure preserving action $\Gamma \curvearrowright (\mathscr{R}, m)$ defined by

$$\forall \gamma \in \Gamma, \forall (y, x) \in \mathscr{R}, \quad \gamma \cdot (y, x) = (\gamma y, x).$$

Set $\mathscr{H} = \mathrm{L}^2(\mathscr{R}, m)$ and define the Koopman unitary representation $\rho : \Gamma \to \mathscr{U}(\mathscr{H})$ by the formula

$$\forall \gamma \in \Gamma, \forall \xi \in \mathscr{H}, \quad (\rho(\gamma)\xi)(y, x) = \xi(\gamma^{-1} y, x).$$

Set $\Delta = \{(x, x) \mid x \in X\} \subset \mathscr{R}$ and observe that $m(\Delta) = 1$. A simple calculation shows that

$$\forall \gamma \in \Gamma, \quad \langle \rho(\gamma) \mathbf{1}_\Delta, \mathbf{1}_\Delta \rangle = \eta(\{x \in X \mid \gamma x = x\}) = \tau(\gamma).$$

It follows that $\pi_\tau \subset \rho$.

Next, we show that ρ is weakly mixing (see [PT13, Proposition 3.1]). It suffices to show that ρ is ergodic (see [AIM19, Proposition 4.5]). It further suffices to show that there is no Γ-invariant measurable subset $\mathscr{W} \subset \mathscr{R}$ such that $0 < m(\mathscr{W}) < +\infty$. Indeed, let $\mathscr{W} \subset \mathscr{R}$ be a Γ-invariant measurable subset such that $m(\mathscr{W}) > 0$. Since $\Gamma\Delta = \mathscr{R}$, we have $m(\mathscr{W} \cap \Delta) > 0$. Define the measurable subset $Y \subset X$ such that $\{(y, y) \mid y \in Y\} = \mathscr{W} \cap \Delta$ and note that $\nu(Y) = m(\mathscr{W} \cap \Delta) > 0$. For ν-almost every $y \in Y$, the orbit Γy is infinite. This further implies that

$$m(\mathscr{W}) \geq \int_Y |\mathscr{W} \cap \{(\gamma x, x) \mid \gamma \in \Gamma\}| \, \mathrm{d}\nu(x) = +\infty \cdot \nu(Y) = +\infty.$$

Since ρ is weakly mixing and $\pi_\tau \subset \rho$, it follows that π_τ is weakly mixing. Theorem B implies that τ is supported on $\mathscr{Z}(\Gamma)$ and so $\mathrm{Stab}_\Gamma(x) \subset \mathscr{Z}(\Gamma)$ for η-almost every $x \in X$.

Let now $\nu \in \mathrm{Prob}_\Gamma(\mathrm{Sub}(\Gamma))$ be an ergodic IRS. By [7s12, Theorem 2.6], there exists an ergodic pmp action $\Gamma \curvearrowright (X, \eta)$ such that $\nu = \mathrm{Stab}_* \eta$. Then the previous paragraph implies that $\mathrm{supp}(\nu) \subset \mathrm{Sub}(\mathscr{Z}(\Gamma))$ and so $\mathrm{supp}(\nu)$ is finite.

(iii) Let $X \subset \mathrm{Sub}(\Gamma)$ be a URS. Consider the restricted Γ-equivariant affine continuous map $\beta|_{\mathrm{Prob}(X)} : \mathrm{Prob}(X) \to \mathscr{P}(\Gamma)$. By Theorem A, the nonempty Γ-invariant weak-$*$ compact convex subset $\beta(\mathrm{Prob}(X)) \subset \mathscr{P}(\Gamma)$ contains a trace τ. Choose $\nu \in \mathrm{Prob}(X)$ so that $\beta(\nu) = \tau$. Then we have $\tau(\gamma) = \nu(\{H \in X \mid \gamma \in H\})$ for every $\gamma \in \Gamma$. Observe that we have the following inclusion of unitary representations

$$(\pi_\tau, \mathscr{H}_\tau, \xi_\tau) \subset \int_X^{\oplus} (\lambda_{\Gamma/H}, \ell^2(\Gamma/H), \delta_H) \, \mathrm{d}\nu(H) = (\rho, \mathscr{K}, \zeta).$$

If τ is supported on $\mathscr{Z}(\Gamma)$, then we have $H \subset \mathscr{Z}(\Gamma)$ for ν-almost every $H \in X$. Since X is a URS, we have $X \subset \mathrm{Sub}(\mathscr{Z}(\Gamma))$ and so X is finite.

If π_τ contains a finite dimensional subrepresentation, then ρ is amenable and so there exists a $\mathrm{Ad}(\rho)$-invariant state $\Psi \in \mathrm{B}(\mathscr{K})^*$. Define the Γ-equivariant unital $*$-homomorphism

$$\Theta : \mathrm{C}(X) \to \mathrm{B}(\mathscr{K}) : F \mapsto \int_X^{\oplus} F_H \, \mathrm{d}\nu(H)$$

where $F_H \in \ell^\infty(\Gamma/H)$ with $F_H(\gamma H) = F(\gamma H \gamma^{-1})$ for every $\gamma \in \Gamma$ and every $H \in X$. Then $\Psi \circ \Theta \in \mathfrak{S}_\Gamma(\mathrm{C}(X))$ is a Γ-invariant state on $\mathrm{C}(X)$ and so there exists a Γ-invariant Borel probability measure $\eta \in \mathrm{Prob}_\Gamma(X)$. Upon considering the ergodic decomposition of $\eta \in \mathrm{Prob}_\Gamma(X)$, we may assume that $\eta \in \mathrm{Prob}_\Gamma(X)$ is an ergodic Γ-invariant Borel probability measure. By item (ii), we have that $X = \mathrm{supp}(\eta)$ is finite. □

Combining Theorems A and B, we also obtain novel results regarding the simplicity and the unique trace property for the C^*-algebra $\mathrm{C}^*_\pi(\Gamma)$ associated with an arbitrary nonamenable (resp. weakly mixing) unitary representation $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$. In particular, Theorem D below provides a far reaching generalization of the results obtained by Bekka–Cowling–de la Harpe [BCH94] for the reduced C^*-algebra $\mathrm{C}^*_\lambda(\Gamma)$. We also refer to [KK14, BKKO14, Ha15, Ke15] for the simplicity and the unique trace property for the reduced C^*-algebra $\mathrm{C}^*_\lambda(\Lambda)$ of countable discrete groups.

Theorem D ([BH19, BBHP20])**.** *Let $\Gamma < G$ be a higher rank lattice. Let $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$ be a unitary representation. Then $\mathrm{C}^*_\pi(\Gamma)$ admits a trace.*

Assume moreover that G has trivial center. If π is not amenable, then $\lambda \prec \pi$ and the unique unital $*$-homomorphism $\Theta : \mathrm{C}^*_\pi(\Gamma) \to \mathrm{C}^*_\lambda(\Gamma) : \pi(\gamma) \mapsto \lambda(\gamma)$ satisfies the following properties:

(i) *$\tau = \tau_\Gamma \circ \Theta$ is the unique trace on $\mathrm{C}^*_\pi(\Gamma)$ and $\ker(\Theta)$ is the unique proper maximal ideal of $\mathrm{C}^*_\pi(\Gamma)$.*

(ii) *τ satisfies the following Powers averaging property*

$$\forall x \in \mathrm{C}^*_\pi(\Gamma), \quad \tau(x)1 \in \overline{\mathrm{conv}} \left\{ \pi(\gamma) x \pi(\gamma)^* \mid \gamma \in \Gamma \right\}.$$

In case G has property (T), the above properties hold as soon as π does not contain any nonzero finite dimensional subrepresentation.

Proof of Theorem D. Let $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$ be a unitary representation and set $A = \mathrm{C}^*_\pi(\Gamma)$. We may extend π to a surjective unital $*$-homomorphism $\pi : \mathrm{C}^*(\Gamma) \to A$ and we may regard $\mathfrak{S}(A) \subset \mathscr{P}(\Gamma)$ as a Γ-invariant weak-$*$ compact convex subset via the Γ-equivariant continuous injective mapping $\mathfrak{S}(A) \hookrightarrow \mathscr{P}(\Gamma) : \psi \mapsto \psi \circ \pi$. Then Theorem A implies that the C^*-algebra A admits a trace.

(i) Assume moreover that G has trivial center and that π is not amenable. Let $\tau \in \mathfrak{S}(A)$ be a trace on A that we regard as a trace $\varphi = \tau \circ \pi \in \mathscr{T}(\Gamma)$ on Γ. We may extend the GNS representation $\pi_\varphi : \mathrm{C}^*(\Gamma) \to \mathrm{C}^*_{\pi_\varphi}(\Gamma)$. Since $\ker(\pi) \subset \ker(\pi_\varphi)$, we have $\pi_\varphi \prec \pi$ and so π_φ is not amenable. Theorem B implies that $\tau \circ \pi = \varphi = \delta_e$. Then $\pi_\varphi = \lambda$ is the left regular representation and $\lambda \prec \pi$. Denote by $\Theta : \mathrm{C}^*_\pi(\Gamma) \to \mathrm{C}^*_\lambda(\Gamma)$ the unique $*$-homomorphism such that $\Theta \circ \pi = \lambda$. Then $\tau_\Gamma \circ \Theta = \tau$ is the unique trace on $A = \mathrm{C}^*_\pi(\Gamma)$. Let $J \lhd A$ be a proper ideal and define $\rho : A \to A/J$ the quotient $*$-homomorphism. Then $\rho \circ \pi \prec \pi$ and so $\rho \circ \pi$ is not amenable. The previous reasoning implies that $\lambda \prec \rho \circ \pi$ and so there is a $*$-homomorphism $\overline{\Theta} : A/J \to \mathrm{C}^*_\lambda(\Gamma)$ such that $\Theta = \overline{\Theta} \circ \rho$. Then $J = \ker(\rho) \subset \ker(\Theta)$. Therefore, $\ker(\Theta) \lhd \mathrm{C}^*_\pi(\Gamma)$ is the unique maximal proper ideal.

(ii) The previous reasoning shows that any nonempty Γ-invariant weak-$*$ compact convex subset $\mathscr{C} \subset \mathfrak{S}(A)$ contains a trace τ such that $\tau \circ \pi = \delta_e$.

Claim 1.1. For every $n \geq 1$, any nonempty Γ-invariant weak-$*$ compact convex subset $\mathscr{C} \subset \mathfrak{S}(A)^n$ contains the fixed point $\tau^{(n)} = (\tau, \dots, \tau)$.

We prove the claim by induction over $n \geq 1$. It is true for $n = 1$ as we already explained. Assume that it is true for $n \geq 1$ and let us prove that it is true for $n + 1$. Let $\mathscr{C} \subset \mathfrak{S}(A)^{n+1}$ be a nonempty Γ-invariant weak-$*$ compact convex subset. The image of $\mathscr{C}$ under the projection map $\mathscr{C}^{n+1} \to \mathscr{C} : (c_1, \dots, c_{n+1}) \mapsto c_1$ contains the trace τ by applying the claim to the case $n = 1$. Then we can identify $\mathscr{C} \cap \{\tau\} \times \mathfrak{S}(A)^n$ with a nonempty Γ-invariant weak-$*$ compact convex subset of $\mathfrak{S}(A)^n$. By applying the claim to the case n, we have $\tau^{(n+1)} \in \mathscr{C} \cap \{\tau\} \times \mathfrak{S}(A)^n$. This finishes the proof of the claim.

Fix a countable weak-$*$ dense subset $\{\psi_n \mid n \geq 1\}$ in $\mathfrak{S}(A)$ and an increasing family of finite subsets $\mathscr{F}_n \subset \Gamma$ such that $\mathscr{F}_1 = \{e\}$ and $\bigcup_{n \geq 1} \mathscr{F}_n = \Gamma$.

Claim 1.2. For every $n \geq 1$, there exists $\mu_n \in \mathrm{Prob}(\Gamma)$ such that

$$\forall 1 \leq k \leq n, \forall \gamma \in \mathscr{F}_n, \quad |(\mu_n * \psi_k - \tau)(\pi(\gamma))| \leq \frac{1}{n}. \tag{1.1}$$

Indeed, for every $n \geq 1$, consider the weak-$*$ closure $\mathscr{C} \subset \mathfrak{S}(A)^n$ of the convex hull of the set $\Gamma \cdot (\psi_1, \dots, \psi_n)$. Since $\tau^{(n)} \in \mathscr{C}$, there exists $\mu_n \in \mathrm{Prob}(\Gamma)$ such that (1.1) holds.

Since $\{\psi_n \mid n \geq 1\}$ is weak-$*$ dense in $\mathfrak{S}(A)$, (1.1) implies that for every $\psi \in \mathfrak{S}(A)$, we have $\mu_n * \psi \to \tau$ for the weak-$*$ topology as $n \to \infty$. Then Hahn–Banach theorem implies that for every $x \in A$, we have $\tau(x)1 \in \overline{\mathrm{conv}} \{\pi(\gamma) x \pi(\gamma)^* \mid \gamma \in \Gamma\}$. □

1.4. The noncommutative Margulis' factor theorem and Connes' rigidity conjecture.

The following outstanding conjecture by Connes concerning the isomorphism class of the group von Neumann algebra $\mathrm{L}(\Gamma)$ of higher rank lattices $\Gamma < G$ can be regarded as a noncommutative analogue of Mostow–Margulis' strong rigidity theorem [Mo73, Ma91]

Connes' rigidity conjecture for higher rank lattices. For every $i \in \{1, 2\}$, let $\Gamma_i < G_i$ be a higher rank lattice with trivial center. If $\mathrm{L}(\Gamma_1) \cong \mathrm{L}(\Gamma_2)$, then $G_1 \cong G_2$ and in particular $\mathrm{rk}_\mathbb{R}(G_1) = \mathrm{rk}_\mathbb{R}(G_2)$.

To put Connes' rigidity conjecture into context, let us mention that over the last two decades, there has been tremendous progress in the classification of group von Neumann algebras thanks to Popa's deformation/rigidity theory (see the ICM surveys [Po06, Va10, Io18]).

Let $\Gamma < G$ be a higher rank lattice. We denote by $P < G$ a minimal parabolic subgroup and whenever $P < Q < G$ is an intermediate parabolic subgroup, we consider the canonical factor map $p_Q : G/P \to G/Q : gP \mapsto gQ$ and we regard $\mathrm{L}(\Gamma \curvearrowright G/Q) \subset \mathrm{L}(\Gamma \curvearrowright G/P)$ as a von Neumann subalgebra.

We recently obtained the following noncommutative analogue of Margulis' factor theorem.

Theorem E ([BH22]). *Let $\Gamma < G$ be a higher rank lattice and assume that G has trivial center. Let $\mathrm{L}(\Gamma) \subset \mathscr{M} \subset \mathrm{L}(\Gamma \curvearrowright G/P)$ be an intermediate von Neumann subalgebra. Then there is a unique intermediate parabolic subgroup $P < Q < G$ such that*

$$\mathscr{M} = \mathrm{L}(\Gamma \curvearrowright G/Q).$$

It is well-known that there are exactly $2^{\mathrm{rk}_{\mathbb{R}}(G)}$ intermediate parabolic subgroups $P < Q < G$. Thus, Theorem E implies that there are exactly $2^{\mathrm{rk}_{\mathbb{R}}(G)}$ intermediate von Neumann subalgebras $\mathrm{L}(\Gamma) \subset \mathscr{M} \subset \mathrm{L}(\Gamma \curvearrowright G/P)$ and so the real rank $\mathrm{rk}_{\mathbb{R}}(G)$ is an invariant of the inclusion $\mathrm{L}(\Gamma) \subset \mathrm{L}(\Gamma \curvearrowright G/P)$.

Theorem E provides strong evidence towards Connes' rigidity conjecture for higher rank lattices. Indeed, Theorem E suggests that in order to prove that the real rank $\mathrm{rk}_{\mathbb{R}}(G)$ is an invariant of the group von Neumann algebra $\mathrm{L}(\Gamma)$, one needs to better understand the inclusion $\mathrm{L}(\Gamma) \subset \mathrm{L}(\Gamma \curvearrowright G/P)$.

2. Lecture 2: The noncommutative Nevo–Zimmer theorem and applications

In the second lecture, we recall Furstenberg's boundary theory of semisimple Lie groups and lattices. We introduce the concept of boundary structures in the von Neumann algebraic framework and we state the noncommutative Nevo–Zimmer theorem for actions of higher rank lattices on von Neumann algebras due to Boutonnet–Houdayer [BH19]. From this, we derive the main results stated in the first lecture.

2.1. **Poisson boundaries.** Let H be a locally compact second countable (lcsc) group. We say that a Borel probability measure $\mu \in \mathrm{Prob}(H)$ is *admissible* if the following conditions are satisfied:

(i) μ is absolutely continuous with respect to the Haar measure;
(ii) $\mathrm{supp}(\mu)$ generates H as a semigroup;
(iii) $\mathrm{supp}(\mu)$ contains a neighborhood of the identity element $e \in H$.

We say that a bounded measurable function $F : H \to \mathbb{C}$ is (right) μ-*harmonic* if

$$\forall g \in H, \quad F(g) = \int_H F(gh)\, \mathrm{d}\mu(h).$$

Any μ-harmonic function is continuous. We denote by $\mathrm{Har}^\infty(H, \mu) \subset \mathrm{C}_b(H)$ the space of all (right) μ-harmonic functions. The left translation action $\lambda : H \curvearrowright \mathrm{C}_b(H)$ leaves the subspace $\mathrm{Har}^\infty(H, \mu)$ globally invariant.

Let (X, ν) be a standard probability space endowed with a measurable action $H \curvearrowright X$. We say that (X, ν) is a (H, μ)-*space* if ν is μ-*stationary*, that is, $\mu * \nu = \nu$. For any (H, μ)-space (X, ν), define the *Poisson transform* $\Psi_\mu : \mathrm{L}^\infty(X, \nu) \to \mathrm{Har}^\infty(H, \mu)$ by the formula

$$\forall f \in \mathrm{L}^\infty(X, \nu), \forall g \in H, \quad \Psi_\mu(f)(g) = \int_X f(gx)\, \mathrm{d}\nu(x).$$

The mapping $\Psi_\mu : \mathrm{L}^\infty(X, \nu) \to \mathrm{Har}^\infty(H, \mu)$ is H-equivariant, unital, positive and contractive.

Theorem 2.1 (Furstenberg, [Fu62b])**.** *There exists a unique (H, μ)-space (B, ν_B) for which the Poisson transform $\Psi_\mu : \mathrm{L}^\infty(B, \nu_B) \to \mathrm{Har}^\infty(H, \mu)$ is bijective.*

The (H, μ)-space (B, ν_B) is called the (H, μ)-*Poisson boundary*. For a construction of the (H, μ)-space (B, ν_B), we also refer to [BS04, Fu00]. The (H, μ)-space (B, ν_B) enjoys remarkable ergodic theoretic properties. In that respect, let $(E, \|\cdot\|)$ be a separable continuous isometric Banach H-module and $\mathscr{C} \subset E^*$ a nonempty H-invariant weak-$*$ compact convex subset. Denote by $\mathrm{Bar} : \mathrm{Prob}(\mathscr{C}) \to \mathscr{C}$ the H-equivariant continuous affine barycenter map. A point $c \in \mathscr{C}$ is μ-*stationary* if $\mathrm{Bar}(\iota_{c*}\mu) = c$ where $\iota_c : H \to \mathscr{C} : g \mapsto gc$

is the orbit map associated with $c \in \mathscr{C}$. By Markov–Kakutani's fixed point theorem, the subset $\mathscr{C}_\mu \subset \mathscr{C}$ of all μ-stationary points in $\mathscr{C}$ is not empty.

The following theorem due to Furstenberg provides the existence and uniqueness of boundary maps (see also [BS04, Section 2]).

Theorem 2.2 (Furstenberg, [Fu62b])**.** *Let $c \in \mathscr{C}_\mu$ be a μ-stationary point. Then there exists an essentially unique H-equivariant measurable map $\beta : B \to \mathscr{C}$ such that*

$$\mathrm{Bar}(\beta_* \nu_B) = c.$$

We say that $\beta : B \to \mathscr{C}$ is the H-equivariant boundary map *associated with $c \in \mathscr{C}_\mu$.*

2.2. **Semisimple Lie groups.** Let G be a connected semisimple real Lie group with finite center and no nontrivial compact factors. Fix an Iwasawa decomposition $G = KAV$, where $K < G$ is a maximal compact subgroup, $A < G$ is a Cartan subgroup and $V < G$ is a unipotent subgroup. Denote by $L = \mathscr{Z}_G(A)$ the centralizer of A in G and set $P = LV$. Then $P < G$ is a minimal parabolic subgroup. Since $K \curvearrowright G/P$ is transitive, G/P is a compact homogeneous space and there exists a unique K-invariant Borel probability measure $\nu_P \in \mathrm{Prob}(G/P)$. The measure class of ν_P coincides with the unique G-invariant measure class on G/P. More generally, for every parabolic subsgroup $P < Q < G$, we denote by $\nu_Q \in \mathrm{Prob}(G/Q)$ the unique K-invariant Borel probability measure on G/Q.

Example 2.3. Assume that $G = \mathrm{SL}_d(\mathbb{R})$ for $d \geq 2$. Then we may take $K = \mathrm{SO}_d(\mathbb{R})$, $A < G$ the subgroup of diagonal matrices with positive entries and $V < G$ the subgroup of strict upper triangular matrices. In that case, $P = \mathscr{Z}(G)AV < G$ is the subgroup of upper triangular matrices. The homogeneous space G/P is the *full flag variety* which consists of all flags $\{0\} \subset W_1 \subset \cdots \subset W_d = \mathbb{R}^d$, where $W_i \subset \mathbb{R}^d$ is a vector subspace such that $\dim_{\mathbb{R}}(W_i) = i$ for every $1 \leq i \leq d$.

Observe that for any left K-invariant Borel probability measure $\mu_G \in \mathrm{Prob}(G)$, the probability measure $\mu_G * \nu_P$ is K-invariant on G/P and so $\mu_G * \nu_P = \nu_P$, that is, $(G/P, \nu_P)$ is a (G, μ_G)-space. Furstenberg [Fu62a] proved the following fundamental result describing the Poisson boundary of semisimple Lie groups.

Theorem 2.4 (Furstenberg, [Fu62a])**.** *Let $\mu_G \in \mathrm{Prob}(G)$ be a K-invariant admissible Borel probability measure. Then $(G/P, \nu_P)$ is the (G, μ_G)-Poisson boundary.*

For lattices $\Gamma < G$ in connected semisimple *real* Lie groups as above, Furstenberg [Fu67] showed that $(G/P, \nu_P)$ can be regarded as the (Γ, μ_Γ)-Poisson boundary with respect to a well chosen probability measure $\mu_\Gamma \in \mathrm{Prob}(\Gamma)$.

Theorem 2.5 (Furstenberg, [Fu67])**.** *Let $\Gamma < G$ be a lattice. Then there exists a probability measure $\mu_\Gamma \in \mathrm{Prob}(\Gamma)$ with full support such that $(G/P, \nu_P)$ is the (Γ, μ_Γ)-Poisson boundary.*

Proof. We give a short proof following [BV22, Proposition 2.24]. Fix a Haar measure m_G on $G = KAV$ which is then necessarily left K-invariant. We may choose $\psi \in \mathrm{C}_c(G)$ such that $\psi \geq 0$, ψ is left K-invariant, $\int_G \psi \, \mathrm{d}m_G = 1$, $\mathrm{supp}(\psi)$ contains a neighborhood of $e \in G$ and $\mathrm{supp}(\psi)$ generates G as a semigroup. Then $\mu_G = \psi \cdot m_G \in \mathrm{Prob}(G)$ is a left K-invariant admissible Borel probability measure. By Theorem 2.4, $(G/P, \nu_P)$ is the (G, μ_G)-Poisson boundary.

By [Ma91, Proposition VI.4.1], we may choose a probability measure $\mu_\Gamma \in \mathrm{Prob}(\Gamma)$ with full support such that $(G/P, \nu_P)$ is a (Γ, μ_Γ)-space. Moreover, we may choose $\mu_\Gamma \leq \theta$ for $\theta : \Gamma \to (0, 1]$ small enough so that μ_Γ has finite logarithmic first moment and finite random walk entropy. We may now use [Ka97, Theorem 10.7] to infer that $(G/P, \nu_P)$ is the (Γ, μ_Γ)-Poisson boundary. □

We call a probability measure $\mu_\Gamma \in \mathrm{Prob}(\Gamma)$ as in Theorem 2.5 a *Furstenberg measure.* Combining Theorems 2.4 and 2.5, we have

$$\mathrm{Har}^\infty(G, \mu_G) \underset{G\text{-equiv.}}{\cong} \mathrm{L}^\infty(G/P, \nu_P) \underset{\Gamma\text{-equiv.}}{\cong} \mathrm{Har}^\infty(\Gamma, \mu_\Gamma).$$

A combination of Theorem 2.4 (resp. Theorem 2.5) and [GM89] implies that for any intermediate parabolic subgroup $P < Q < G$, the map

$$\delta \circ p_Q : G/P \to \mathrm{Prob}(G/Q) : gP \mapsto \delta_{gQ} \tag{2.1}$$

is the essentially unique Γ-equivariant (resp. G-equivariant) measurable mapping $\zeta : G/P \to \mathrm{Prob}(G/Q)$.

2.3. **Boundary structures.** For any C^*-algebra $A \subset \mathrm{B}(\mathscr{H})$ and any $n \geq 1$, $\mathrm{M}_n(A) = \mathrm{M}_n(\mathbb{C}) \otimes A \subset \mathrm{B}(\mathscr{H}^{\oplus n})$ is naturally a C^*-algebra. Let A, B be C^*-algebras. A linear map $\Theta : A \to B$ is said to be *unital completely positive* (ucp) if Θ is unital and if for every $n \geq 1$, the linear map

$$\Theta^{(n)} : \mathrm{M}_n(A) \to \mathrm{M}_n(B) : [a_{ij}]_{ij} \mapsto [\Theta(a_{ij})]_{ij}$$

is positive. Any unital $*$-homomorphism $\pi : A \to B$ is a ucp map. When A or B is commutative, any unital positive linear map $\Theta : A \to B$ is automatically ucp (see e.g. [Pa02, Theorems 3.9 and 3.11]).

Let $\Theta : A \to B$ be a ucp map. The following Schwarz inequality holds

$$\forall a \in A, \quad \Theta(a)^* \Theta(a) \leq \Theta(a^* a).$$

Moreover, the subspace $\mathrm{mult}(\Theta) \subset A$ defined by

$$\mathrm{mult}(\Theta) = \{a \in A \mid \Theta(a^* a) = \Theta(a)^* \Theta(a) \text{ and } \Theta(a a^*) = \Theta(a) \Theta(a)^*\}$$

is a C^*-subalgebra. Then $\mathrm{mult}(\Theta) \subset A$ is the largest C^*-subalgebra on which Θ restricts to a unital $*$-homomorphism. If H is a locally compact group, M, N are H-von Neumann algebras and $\Theta : M \to N$ is a H-equivariant normal ucp map, then $\mathrm{mult}(\Theta) \subset M$ is a H-invariant von Neumann subalgebra.

For any inclusion of von Neumann algebras $N \subset M$, a (normal) *conditional expectation* $\mathrm{E} : M \to N$ is a (normal) ucp map such that $\mathrm{E} \circ \mathrm{E} = \mathrm{E}$.

Examples 2.6. We will be using the following examples of normal conditional expectations.

(i) For any von Neumann algebra M endowed with a faithful normal trace τ and any von Neumann subalgebra $B \subset M$, there exists a unique faithful normal conditional expectation $\mathrm{E}_B : M \to B$ such that $\tau \circ \mathrm{E}_B = \tau$.

(ii) For any countable discrete group Λ and any nonsingular action $\Lambda \curvearrowright (X, \eta)$, there exists a canonical Λ-equivariant faithful normal conditional $\mathrm{E} : \mathrm{L}(\Lambda \curvearrowright X) \to \mathrm{L}^\infty(X)$ which satisfies

$$\forall T = \sum_{\gamma \in \Lambda} F_\gamma u_\gamma \in \mathrm{L}(\Lambda \curvearrowright X), \quad \mathrm{E}(T) = F_e.$$

Definition 2.7 ([BBHP20])**.** Let $\Gamma < G$ be a higher rank lattice and $H = \Gamma$ or $H = G$. Let M be a H-von Neumann algebra. A *H-boundary structure* $\Theta : M \to \mathrm{L}^\infty(G/P)$ is a H-equivariant faithful normal ucp map. We simply say that Θ is *invariant* if $\Theta(M) = \mathbb{C}1$.

In the language of ucp maps, (2.1) precisely says that for every intermediate parabolic subgroup $P < Q < G$, the canonical inclusion $\mathrm{C}(G/Q) \hookrightarrow \mathrm{L}^\infty(G/P) : F \mapsto F \circ p_Q$ is the only H-equivariant ucp map $\Theta : \mathrm{C}(G/Q) \to \mathrm{L}^\infty(G/P)$.

In the setting of higher rank lattices in connected semisimple *real* Lie groups, the notion of boundary structure is equivalent to the notion of stationary state. Indeed, for $H = \Gamma$ or $H = G$, fix a Borel probability measure $\mu_H \in \mathrm{Prob}(H)$ so that $(G/P, \nu_P)$ is the (H, μ_H)-Poisson boundary (see Theorems 2.4 and 2.5). We then identify $\mathrm{Har}^\infty(H, \mu_H) \cong \mathrm{L}^\infty(G/P, \nu_P)$ as H-operator systems.

- If $\Theta : M \to \mathrm{L}^\infty(G/P)$ is a H-boundary structure, then $\varphi = \nu_P \circ \Theta \in M_*$ is a faithful normal μ_H-stationary state on M. Moreover, if Θ is invariant, then φ is H-invariant.
- Conversely, let $\varphi \in M_*$ be a faithful normal μ_H-stationary state on M. Define the H-equivariant faithful normal ucp map

$$\Theta : M \to \mathrm{Har}^\infty(H, \mu_H) : x \mapsto \left(h \mapsto \varphi(h^{-1}x)\right).$$

  Since $\mathrm{Har}^\infty(H, \mu_H) \cong \mathrm{L}^\infty(G/P, \nu_P)$ as H-operator systems, we may further regard $\Theta : M \to \mathrm{L}^\infty(G/P)$ as a H-boundary structure such that $\varphi = \nu_P \circ \Theta$. If φ is H-invariant, then Θ is invariant.

Using the above observation and [BBHP20, Proposition 2.7], if the action $\sigma : H \curvearrowright M$ is ergodic, then there is at most one H-boundary structure $\Theta : M \to \mathrm{L}^\infty(G/P)$. In that case, the H-boundary structure $\Theta : M \to \mathrm{L}^\infty(G/P)$ is a canonical object attached to M.

The notion of boundary structure is well adapted to induction. Fix a Borel section $\tau : G/\Gamma \to G$ so that $x = \tau(x)\Gamma$ for every $x \in G/\Gamma$. Consider the Borel 1-cocycle $c : G \times G/\Gamma \to \Gamma : (g, x) \mapsto \tau(gx)^{-1} g\tau(x)$. Let M be a Γ-von Neumann algebra and $\Theta : M \to \mathrm{L}^\infty(G/P)$ a Γ-boundary structure. Consider the *induced von Neumann algebra* $\widehat{M} = \mathrm{Ind}_\Gamma^G(M) \cong \mathrm{L}^\infty(G/\Gamma) \mathbin{\overline{\otimes}} M$ endowed with the continuous action $G \curvearrowright \mathrm{Ind}_\Gamma^G(M)$ defined by

$$\forall g \in G, \forall F \in \mathrm{Ind}_\Gamma^G(M), \quad (g \cdot F)(x) = c(g, x) \cdot F(g^{-1}x).$$

Alternatively, consider the von Neumann algebra $\mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} M$ together with the continuous action $G \times \Gamma \curvearrowright \mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} M$ defined by

$$\forall (g, \gamma) \in G \times \Gamma, \forall F \in \mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} M, \quad ((g, \gamma) \cdot F)(h) = \gamma \cdot F(g^{-1}h\gamma)$$

Then we canonically have $\mathrm{Ind}_\Gamma^G(M) \cong (\mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} M)^\Gamma$ as G-von Neumann algebras. Since G/P is a G-space, we have $\mathrm{Ind}_\Gamma^G(\mathrm{L}^\infty(G/P)) \cong \mathrm{L}^\infty(G/\Gamma) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(G/P)$ as G-von Neumann algebras where $G \curvearrowright G/\Gamma \times G/P$ acts diagonally. Denote by $\nu_\Gamma \in \mathrm{Prob}(G/\Gamma)$ the unique G-invariant Borel probability measure. Then the map $\widehat{\Theta} = \nu_\Gamma \otimes \Theta : \widehat{M} \to \mathrm{L}^\infty(G/P)$ is a G-equivariant faithful normal ucp map. We then refer to $\widehat{\Theta}$ as the *induced G-boundary structure*. Note that Θ is invariant if and only if $\widehat{\Theta}$ is invariant.

This framework provides a conceptual approach to the *stationary induction trick* considered in [BH19, Section 4]. Let $\mu_G \in \mathrm{Prob}(G)$ be a K-invariant admissible Borel probability measure. Let φ be a faithful normal μ_Γ-stationary state on M and define the corresponding Γ-boundary structure $\Theta : M \to \mathrm{L}^\infty(G/P)$ such that $\nu_P \circ \Theta = \varphi$. Consider the induced G-boundary structure $\widehat{\Theta} : \widehat{M} \to \mathrm{L}^\infty(G/P)$. Then $\widehat{\varphi} = \nu_P \circ \widehat{\Theta}$ is a faithful normal μ_G-stationary state on $\widehat{M}$. Moreover, φ is Γ-invariant if and only if $\widehat{\varphi}$ is G-invariant.

Let $\Gamma < G$ be a higher rank lattice and fix a Furstenberg measure $\mu_\Gamma \in \mathrm{Prob}(\Gamma)$. We present below several examples of Γ-boundary structures.

Example 2.8 (Boundary structure arising from topological dynamics)**.** Let $\Gamma \curvearrowright X$ be a minimal action on a compact metrizable space. Choose an extremal μ_Γ-stationary Borel probability measure $\nu \in \mathrm{Prob}_{\mu_\Gamma}(X)$. By minimality, we have $\mathrm{supp}(\nu) = X$. Denote by $\beta : G/P \to \mathrm{Prob}(X) : b \mapsto \beta_b$ the Γ-equivariant boundary map associated with $\nu \in \mathrm{Prob}_{\mu_\Gamma}(X)$ (see Theorem 2.2). By duality, we may consider the Γ-equivariant ucp map $\Theta : \mathrm{C}(X) \to \mathrm{L}^\infty(G/P) : f \mapsto (b \mapsto \beta_b(f))$ which satisfies $\nu_P \circ \Theta = \nu$. By extremality, the nonsingular action $\Gamma \curvearrowright (X, \nu)$ is ergodic. Moreover, $\Theta : \mathrm{C}(X) \to \mathrm{L}^\infty(G/P)$ extends to a Γ-boundary structure $\Theta : \mathrm{L}^\infty(X, \nu) \to \mathrm{L}^\infty(G/P)$.

Example 2.9 (Boundary structure arising from unitary representations)**.** Let $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$ be a unitary representation and set $A = \mathrm{C}^*_\pi(\Gamma)$. Choose an extremal μ_Γ-stationary state $\varphi \in \mathfrak{S}_{\mu_\Gamma}(A)$ and consider the GNS triple $(\pi_\varphi, \mathscr{H}_\varphi, \xi_\varphi)$. Denote by $\beta : G/P \to \mathfrak{S}(A) : b \mapsto \beta_b$ the Γ-equivariant boundary map associated with $\varphi \in \mathfrak{S}_{\mu_\Gamma}(A)$ (see Theorem 2.2). By duality, we may consider the Γ-equivariant ucp map $\Theta : A \to \mathrm{L}^\infty(G/P) : a \mapsto (b \mapsto \beta_b(a))$ which satisfies $\nu_P \circ \Theta = \varphi$. Set $M = \pi_\varphi(A)'' = (\pi_\varphi \circ \pi)(\Gamma)''$. By extremality, the conjugation action $\mathrm{Ad}(\pi_\varphi \circ \pi) : \Gamma \curvearrowright M$ is ergodic. Moreover, the Γ-equivariant ucp map

$$\pi_\varphi(A) \to \mathrm{L}^\infty(G/P) : \pi_\varphi(a) \mapsto \Theta(a)$$

is well defined and extends to a Γ-boundary structure $\Theta : M \to \mathrm{L}^\infty(G/P)$. We refer to [BH19, Proof of Theorem A] for further details.

Example 2.10 (Boundary structure arising from characters)**.** Let $\tau \in \mathscr{C}(\Gamma)$ be a character and denote by $(\pi, \mathscr{H}, \xi)$ its GNS triple. Denote by $J : \mathscr{H} \to \mathscr{H} : \pi(\gamma)\xi \mapsto \pi(\gamma)^*\xi$ the canonical conjugation. Letting $N = \pi(\Gamma)''$, we have $JNJ = N' = \pi(\Gamma)'$. Following [Pe14], define the *noncommutative Poisson boundary* $\mathscr{B}_\tau$ associated with $\tau \in \mathscr{C}(\Gamma)$ as the von Neumann algebra of all ν_P-equivalence classes of essentially bounded measurable functions $f : G/P \to \mathrm{B}(\mathscr{H})$ satisfying $f(\gamma b) = \mathrm{Ad}(J\pi(\gamma)J)(f(b))$ for every $\gamma \in \Gamma$ and almost every $b \in G/P$. Observe that $\mathbb{C}1 \otimes \pi(\Gamma)'' \subset \mathscr{B}_\tau$. Since P is amenable, $\mathscr{B}_\tau \cong (\mathrm{Ind}_\Gamma^G(\mathrm{B}(\mathscr{H})))^P$ is an amenable von Neumann algebra. By extremality, the conjugation action $\mathrm{Ad}(\pi) : \Gamma \curvearrowright \mathscr{B}_\tau$ is ergodic. Moreover,

$$\Theta : \mathscr{B}_\tau \to \mathrm{L}^\infty(G/P) : f \mapsto (b \mapsto \langle f(b)\xi, \xi\rangle)$$

is a Γ-boundary structure. When $\tau = \delta_e$ is the regular character, the noncommutative Poisson boundary $\mathscr{B}_\tau$ coincides with the group measure space von Neumann algebra $\mathrm{L}(\Gamma \curvearrowright G/P)$ and the Γ-boundary structure $\Theta : \mathrm{L}(\Gamma \curvearrowright G/P) \to \mathrm{L}^\infty(G/P)$ is the canonical Γ-equivariant faithful normal conditional expectation. We refer to [BH19, Proof of Theorem C] for further details.

2.4. **The noncommutative Nevo–Zimmer theorem.** The following theorem due to Boutonnet–Houdayer [BH19] is a noncommutative generalization of Nevo–Zimmer's structure theorem for stationary actions of higher rank simple connected real Lie groups with finite center on measure spaces [NZ00, Theorem 1].

Theorem 2.11 (Boutonnet–Houdayer [BH19])**.** *Let $\Gamma < G$ be a higher rank lattice and assume that G is simple with finite center. Let $H = \Gamma$ or $H = G$. Let M be an ergodic H-von Neumann algebra and $\Theta : M \to \mathrm{L}^\infty(G/P)$ an H-boundary structure such that $\Theta(M) \neq \mathbb{C}1$.*

Then there exist a unique proper parabolic subgroup $P < Q < G$ such that $\mathrm{mult}(\Theta) \cong \mathrm{L}^\infty(G/Q)$ *as H-von Neumann algebras.*

Theorem 2.11 extends the work of Nevo–Zimmer [NZ00] in two ways. Firstly, we deal with arbitrary von Neumann algebras M (instead of measure spaces (X, ν)) and secondly, we deal with H-actions $H \curvearrowright M$ with $H = \Gamma$ or $H = G$ (instead of G-actions $G \curvearrowright (X, \nu)$).

Claim 2.12. If Theorem 2.11 holds for $H = G$, then it holds for $H = \Gamma$.

Indeed, assume that Theorem 2.11 holds for $H = G$. Let M be an ergodic Γ-von Neumann algebra and $\Theta : M \to \mathrm{L}^\infty(G/P)$ a Γ-boundary structure such that $\Theta(M) \neq \mathbb{C}1$. Consider the induced G-von Neumann algebra $\widehat{M} = \mathrm{Ind}_\Gamma^G(M)$ together with its induced G-boundary structure $\widehat{\Theta} : \widehat{M} \to \mathrm{L}^\infty(G/P)$. We have $\widehat{\Theta}(\widehat{M}) \neq \mathbb{C}1$. In particular, there exists a proper parabolic $P < Q < G$ and a G-equivariant unital $\ast$-homomorphism $\iota : \mathrm{C}(G/Q) \to \widehat{M}$. We may regard $\widehat{M} = (\mathrm{L}^\infty(G) \mathbin{\overline{\otimes}} M)^\Gamma = \mathrm{L}^\infty(G, M)^\Gamma$. Since the action $G \curvearrowright \mathrm{C}(G/Q)$ is $\|\cdot\|_\infty$-continuous, it follows that $\iota(\mathrm{C}(G/Q)) \subset \mathrm{C}_b(G, M)^\Gamma$. Consider the evaluation $\ast$-homomorphism $\rho : \mathrm{C}_b(G, M)^\Gamma \to M : F \mapsto F(e)$ which is Γ-equivariant. Then $\pi = \rho \circ \iota : \mathrm{C}(G/Q) \to M$ is a Γ-equivariant $\ast$-homomorphism. Then (2.1) implies that π extends to a normal unital $\ast$-homomorphism $\pi : \mathrm{L}^\infty(G/Q) \to M$ such that $\Theta \circ \pi : \mathrm{L}^\infty(G/Q) \hookrightarrow \mathrm{L}^\infty(G/P)$ is the canonical $\ast$-embedding. Then $\iota(\mathrm{L}^\infty(G/Q)) \subset \mathrm{mult}(\Theta)$. Upon considering a smaller proper parabolic subgroup, we have $\mathrm{mult}(\Theta) \cong \mathrm{L}^\infty(G/Q)$. This finishes the proof of the claim.

The remarkable feature of Theorem 2.11 is that when $\Theta : M \to \mathrm{L}^\infty(G/P)$ is not invariant, there is a nontrivial Γ-invariant *commutative* von Neumann subalgebra $M_0 \subset M$ such that $M_0 \cong \mathrm{L}^\infty(G/Q)$. This allows us to exploit the dynamical properties of the ergodic action $\Gamma \curvearrowright G/Q$. In that respect, we record the following well-known fact.

Lemma 2.13. *Let $\Gamma < G$ be a higher rank lattice and assume that G is simple with finite center. For every proper parabolic subgroup $P < Q < G$ and every $\gamma \in \Gamma \setminus \mathscr{Z}(\Gamma)$, the measurable subset $\mathrm{Fix}_{G/Q}(\gamma) = \{c \in G/Q \mid \gamma c = c\}$ is null in G/Q.*

Proof. Observe that $\mathscr{Z}(\Gamma) < \mathscr{Z}(G) < P$. Upon replacing G by $G/\mathscr{Z}(G)$ and Γ by $\Gamma\mathscr{Z}(G)/\mathscr{Z}(G)$, without loss of generality, we may assume that $\Gamma < G$ is a higher rank lattice where G is simple with trivial center.

Since G is a noncompact simple connected real Lie group with trivial center, there exists a simple connected algebraic $\mathbb{R}$-group $\mathbf{G}$ such that $G \cong \mathbf{G}_\mathbb{R}^0$ as Lie groups (see e.g. [Zi84, Proposition 3.1.6]). Note that $\mathbf{G}_\mathbb{R}^0 \lhd \mathbf{G}_\mathbb{R}$ is normal and has finite index. For every proper parabolic subgroup $P < Q < G$, there exists a unique proper parabolic $\mathbb{R}$-subgroup $\mathbf{P} < \mathbf{Q} < \mathbf{G}$ such that $Q = G \cap \mathbf{Q}_\mathbb{R}$ (see [Bo91, Theorem 21.15]). Since $\mathbf{G}_\mathbb{R} = G \cdot \mathbf{P}_\mathbb{R}$, the action $G \curvearrowright \mathbf{G}_\mathbb{R}/\mathbf{P}_\mathbb{R}$ is transitive and so is $G \curvearrowright \mathbf{G}_\mathbb{R}/\mathbf{Q}_\mathbb{R}$ (see [Ma91, Proposition I.1.5.4]). Therefore, we have $G/Q \cong \mathbf{G}_\mathbb{R}/\mathbf{Q}_\mathbb{R}$ as G-spaces.

Denote by $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{Q}$ the canonical algebraic $\mathbb{R}$-morphism. Regard $\mathbf{G}_\mathbb{R}/\mathbf{Q}_\mathbb{R} = \mathbf{G}_\mathbb{R}\pi(e) \subset (\mathbf{G}/\mathbf{Q})_\mathbb{R}$. Let $\gamma \in \Gamma \setminus \{e\}$. Then the fixed-point subset $\mathbf{W} = \{w \in \mathbf{G}/\mathbf{Q} \mid \gamma w = w\}$ is a proper $\mathbb{R}$-subvariety of $\mathbf{G}/\mathbf{Q}$. It follows that $\mathbf{V} = \pi^{-1}(\mathbf{W})$ is a proper algebraic $\mathbb{R}$-subvariety of $\mathbf{G}$. Since $\mathbf{G}$ is connected, [Ma91, Proposition I.2.5.3(ii)] implies that $\mathbf{V}_\mathbb{R}$ is null in $\mathbf{G}_\mathbb{R}$. This further implies that

$$\mathrm{Fix}_{G/Q}(\gamma) = \{c \in G/Q \mid \gamma c = c\} = \pi(\mathbf{V}_\mathbb{R}) \cap \mathbf{G}_\mathbb{R}/\mathbf{Q}_\mathbb{R} = \pi(\mathbf{V}_\mathbb{R}) \cap G/Q$$

is null in G/Q. □

The following consequence of Theorem 2.11 will be crucial in deducing the main applications stated in the first lecture.

Proposition 2.14. *Let $\Gamma < G$ be a higher rank lattice and assume that G is simple with finite center. Let $\pi : \Gamma \to \mathscr{U}(M)$ be a group homomorphism such that $\pi(\Gamma)' \cap M = \mathbb{C}1$. Consider the conjugation action $\mathrm{Ad}(\pi) : \Gamma \curvearrowright M$.*

Let $\Theta : M \to \mathrm{L}^\infty(G/P)$ be a Γ-boundary structure such that $\Theta(M) \neq \mathbb{C}1$. Then we have $\Theta(\pi(\gamma)) = 0$ for every $\gamma \in \Gamma \setminus \mathscr{Z}(\Gamma)$.

Proof. By Theorem 2.11, there exists a unique proper parabolic subgroup $P < Q < G$ such that $\mathrm{mult}(\Theta) \cong \mathrm{L}^\infty(G/Q)$ as Γ-von Neumann algebras. Denote by $p : G/P \to G/Q$ the canonical factor map. Upon identifying $\mathrm{mult}(\Theta) = \mathrm{L}^\infty(G/Q)$ and using (2.1), the restriction $\Theta|_{\mathrm{L}^\infty(G/Q)} : \mathrm{L}^\infty(G/Q) \hookrightarrow \mathrm{L}^\infty(G/P) : F \mapsto F \circ p$ is the canonical inclusion. Denote by $A \subset M$ the separable unital Γ-C*-subalgebra generated by $\pi(\Gamma)$ and $\mathrm{C}(G/Q)$. Then denote by $\beta : G/P \to \mathfrak{S}(A)$ the unique Γ-equivariant measurable map such that $\beta_b(a) = \Theta(a)(b)$ for every $a \in A$ and almost every $b \in G/P$. Then we have $\beta_b|_{\mathrm{C}(G/Q)} = \delta_{p(b)}$ for almost every $b \in G/P$.

Let $\gamma \in \Gamma \setminus \mathscr{Z}(\Gamma)$. By Lemma 2.13, the measurable subset

$$Z = \{c \in G/Q \mid \gamma c \neq c\} \subset G/Q$$

has full measure in G/Q. Then $Y = p^{-1}(Z) \subset G/P$ has full measure in G/P. For every $b \in Y$, we may choose a continuous function $F \in \mathrm{C}(G/Q)$ such that $0 \leq F \leq 1$, $\beta_{\gamma b}(F) = F(p(\gamma b)) = 1$ and $\beta_b(F) = F(p(b)) = 0$. Write $\beta_b(\pi(\gamma)) = \beta_b(\pi(\gamma)F) + \beta_b(\pi(\gamma)(1-F))$. By Cauchy–Schwarz inequality, we have

$$|\beta_b(\pi(\gamma)F)|^2 \leq \beta_b(F^2) = F^2(p(b)) = 0$$

and

$$|\beta_b(\pi(\gamma)(1-F))|^2 = |\beta_{\gamma b}((1-F)\pi(\gamma))|^2 \leq \beta_{\gamma b}((1-F)^2) = 0.$$

Therefore, for every $b \in Y$, we have $\beta_b(\pi(\gamma)) = 0$ and so $\Theta(\pi(\gamma)) = 0$. □

2.5. **Applications to the dynamics of positive definite functions.** Applying Theorem 2.11, we give a proof of Theorems A and B in the case when G is a higher rank simple connected real Lie group with finite center.

Proof of Theorem A. Denote by $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$ the *universal* unitary representation that is equal to the orthogonal direct sum of all cyclic unitary representations. Then $A = \mathrm{C}^*_\pi(\Gamma)$ coincides with the full C*-algebra $\mathrm{C}^*(\Gamma)$ and we may use the identification $\mathfrak{S}(A) = \mathscr{P}(\Gamma)$. Let $\mathscr{C} \subset \mathfrak{S}(A)$ be a nonempty Γ-invariant weak-$*$ compact convex subset. By Markov–Kakutani's fixed point theorem, the subset $\mathscr{C}_{\mu_\Gamma} \subset \mathscr{C}$ of all μ_Γ-stationary states in $\mathscr{C}$ in not empty. We claim that $\mathscr{C}_{\mu_\Gamma} \subset \mathscr{T}(\Gamma)$. By Krein–Milman theorem, it suffices to show that any extreme point $\varphi \in \mathscr{C}_{\mu_\Gamma}$ is Γ-invariant. Let $\varphi \in \mathscr{C}_{\mu_\Gamma}$ be an extreme point and set $M = \pi_\varphi(A)'' = (\pi_\varphi \circ \pi)(\Gamma)''$. Consider the Γ-boundary structure $\Theta : M \to \mathrm{L}^\infty(G/P)$ such that $\nu_P \circ \Theta = \varphi$ as in Example 2.9. If Θ is invariant, then $\varphi \in \mathscr{T}(\Gamma)$. If Θ is not invariant, then

Proposition 2.14 implies that for every $\gamma \in \Gamma \setminus \mathscr{Z}(\Gamma)$, we have $\Theta(\pi(\gamma)) = 0$ and so $\varphi(\gamma) = 0$. Then φ is supported on $\mathscr{Z}(\Gamma)$ and so $\varphi \in \mathscr{T}(\Gamma)$. Therefore, we have $\mathscr{C} \cap \mathscr{T}(\Gamma) \neq \emptyset$. $\square$

Proof of Theorem B. Let $\tau \in \mathscr{C}(\Gamma)$ be a character. Denote by $(\pi, \mathscr{H}, \xi)$ its GNS triple and by $\mathscr{B}_\tau$ its associated noncommutative Poisson boundary. Consider the Γ-boundary structure $\Theta : \mathscr{B}_\tau \to \mathrm{L}^\infty(G/P)$ as in Example 2.10. If Θ is invariant, then for every $f \in \mathscr{B}_\tau$, the function $\Theta(f) : G/P \to \mathbb{C} : b \mapsto \langle f(b)\xi, \xi\rangle$ is (essentially) constant. Since $\mathbb{C}1 \otimes \pi(\Gamma)'' \subset \mathscr{B}_\tau$ and since the linear span of $\pi(\Gamma)\xi$ is dense in $\mathscr{H}$, it easily follows that every $f \in \mathscr{B}_\tau$ is (essentially) constant as a function $G/P \to \mathrm{B}(\mathscr{H})$. This further implies that $\mathscr{B}_\tau = \mathbb{C}1 \otimes \pi(\Gamma)''$ and so $\pi(\Gamma)''$ is amenable. This further implies that π is amenable. Since G is a higher rank connected simple Lie group with finite center, G has property (T) by Kazhdan's theorem, and so Γ has property (T). Since π is amenable, π necessarily contains a nonzero finite dimensional subrepresentation. Thus, π is finite dimensional since $\tau \in \mathscr{C}(\Gamma)$ is a character. In that case, $\tau \in \mathscr{C}(\Gamma)$ is a compact character. If Θ is not invariant, then Proposition 2.14 implies that for every $\gamma \in \Gamma \setminus \mathscr{Z}(\Gamma)$, we have $\Theta(\pi(\gamma)) = 0$ and so $\tau(\gamma) = 0$. Thus, τ is supported on $\mathscr{Z}(\Gamma)$.

Since Γ has property (T), for every $d \geq 1$, there are only finitely many d-dimensional irreducible Γ-unitary representations up to unitary conjugacy. Combining this fact together with the results from the previous paragraphs, we infer that $\mathscr{C}(\Gamma)$ is a countable set. Write $\mathscr{C}(\Gamma) = \{\tau_n \mid n \in \mathbb{N}\}$.

Let now $\tau \in \mathscr{T}(\Gamma)$. There exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $[0, 1]$ such that $\sum_{n\in\mathbb{N}} \alpha_n = 1$ and $\tau = \sum_{n\in\mathbb{N}} \alpha_n \tau_n$. Assume that τ is not supported on $\mathscr{Z}(\Gamma)$. Then there exists $n \in \mathbb{N}$ such that $\alpha_n \neq 0$ and $\tau_n \in \mathscr{C}(\Gamma)$ is a compact character. Since $\tau = \alpha_n\tau_n + \sum_{k\neq n} \alpha_k \tau_k$, we have $\pi_{\tau_n} \subset \pi_\tau$ and so π_τ contains a nonzero finite dimensional subrepresentation. $\square$

2.6. **Proof of the noncommutative Margulis' factor theorem.** Applying Theorem 2.11, we give a proof of Theorem E in the case when G is a higher rank simple connected real Lie group with trivial center.

Proof of Theorem E. Set $\mathscr{B} = \mathrm{L}(\Gamma \curvearrowright G/P)$. By [BH22, Lemma 3.2], the conjugation action $\Gamma \curvearrowright \mathscr{B}$ is ergodic. Denote by $\mathrm{E} : \mathscr{B} \to \mathrm{L}^\infty(G/P)$ the canonical Γ-equivariant conditional expectation. Let $\mathrm{L}(\Gamma) \subset \mathscr{M} \subset \mathscr{B}$ be an intermediate von Neumann subalgebra and consider the Γ-boundary structure $\Theta = \mathrm{E}|_{\mathscr{M}} : \mathscr{M} \to \mathrm{L}^\infty(G/P)$.

Firstly, assume that Θ is invariant. We simply denote by $u_\gamma \in \mathrm{L}(\Gamma) \subset \mathscr{B}$ the canonical unitaries implementing the action $\Gamma \curvearrowright G/P$. For every $x \in M$, write $x = \sum_{\gamma\in\Gamma} x_\gamma u_\gamma$ for its Fourier expansion, where $x_\gamma = \mathrm{E}(xu_\gamma^*)$ for every $\gamma \in \Gamma$. Since $\mathrm{E}|_M = \Theta$ is invariant and since $\mathrm{L}(\Gamma) \subset M$, it follows that $x_\gamma \in \mathbb{C}1$ for every $\gamma \in \Gamma$ and every $x \in M$. Then [Su18, Corollary 3.4] implies that $M = \mathrm{L}(\Gamma)$.

Secondly, assume that Θ is not invariant. By Theorem 2.11, there exist a proper parabolic subgroup $P < Q < G$ and a Γ-equivariant normal unital embedding $\iota : \mathrm{L}^\infty(G/Q) \hookrightarrow M$ such that $\mathrm{E} \circ \iota : \mathrm{L}^\infty(G/Q) \hookrightarrow \mathrm{L}^\infty(G/P)$ is the canonical inclusion. This further implies that $\mathrm{L}(\Gamma \curvearrowright G/Q) = \mathrm{L}(\Gamma) \vee \mathrm{L}^\infty(G/Q) \subset M$. Since the nonsingular action $\Gamma \curvearrowright (G/Q, \nu_Q)$ is essentially free by Lemma 2.13, a combination of [Su18, Theorem 3.6] and [Ma91, Theorem IV.2.11] implies that there exists a parabolic subgroup $P < R < Q$ such that $M = \mathrm{L}(\Gamma \curvearrowright G/R)$. □

3. Lecture 3: Proof of the noncommutative Nevo–Zimmer theorem

In the third lecture, we prove the noncommutative Nevo–Zimmer theorem for ergodic actions of higher rank connected simple real Lie groups with finite center on von Neumann algebras due to Boutonnet–Houdayer [BH19].

3.1. **Statement of the main result.** We will prove the following noncommutative analogue of Nevo–Zimmer's structure theorem for faithful stationary ergodic actions of higher rank connected simple real Lie groups with finite center (see [NZ00, Theorem 1]).

Theorem 3.1 (Boutonnet–Houdayer [BH19])**.** *Let G be a higher rank simple connected real Lie group with finite center. Let $\mathscr{M}$ be an ergodic G-von Neumann algebra and $\Theta : \mathscr{M} \to \mathrm{L}^\infty(G/P)$ a G-boundary structure such that $\Theta(\mathscr{M}) \neq \mathbb{C}1$.*

Then there exists a unique proper parabolic subgroup $P < Q < G$ such that $\mathrm{mult}(\Theta) \cong \mathrm{L}^\infty(G/Q)$ as G-von Neumann algebras.

A few remarks are in order. In the case when $Q = P$, the restriction $\Theta|_{\mathrm{mult}(\Theta)} : \mathrm{mult}(\Theta) \to \mathrm{L}^\infty(G/P)$ is a normal unital surjective $*$-isomorphism and so we may regard $\mathrm{L}^\infty(G/P) \subset \mathscr{M}$ as a G-invariant von Neumann subalgebra and $\Theta : \mathscr{M} \to \mathrm{L}^\infty(G/P)$ as a G-equivariant faithful normal conditional expectation.

Fix a maximal compact subgroup $K < G$. Assume that $\mathscr{M} = \mathrm{L}^\infty(X, \nu)$ where (X, ν) is a faithful ergodic (G, μ_G)-space where $\mu_G \in \mathrm{Prob}(G)$ is a left K-invariant admissible Borel probability measure. Denote by $\Theta : \mathscr{M} \to \mathrm{L}^\infty(G/P)$ the corresponding Poisson transform. In that case, Theorem 3.1 is equivalent to [NZ00, Theorem 1]. If ν is not G-invariant, then the proper parabolic subgroup $P < Q < G$ appearing in Theorem 3.1 corresponds to the maximal projective factor $(X, \nu) \to (G/Q, \nu_Q)$ considered in [NZ00, Lemma 0.1].

The proof of Theorem 3.1 consists of two independent steps.

- The first step of the proof consists of reducing the problem to a commutative setting where the group action is still faithful and moreover possesses large stabilizers (see Theorem 3.3).
- The second step of the proof due to Nevo–Zimmer [NZ00] consists of proving that faithful stationary ergodic actions with large stabilizers possess a nontrivial projective factor.

We will prove the following more general version of the noncommutative Nevo–Zimmer structure theorem for higher rank simple algebraic groups defined over a local field of characteristic zero (see [BH19, Theorem 5.1] for the case of Lie groups and [BBH21, Theorem 1.5] for the case of algebraic groups defined over a local field of arbitrary characteristic).

Theorem 3.2. *Let k be a local field of characteristic zero. Let $\mathbf{G}$ be a simple connected algebraic k-group such that $\mathrm{rk}_k(\mathbf{G}) \geq 2$ and fix a minimal parabolic k-subgroup $\mathbf{P} < \mathbf{G}$. Let $\mathscr{M}$ be an ergodic $\mathbf{G}_k$-von Neumann algebra and $\Theta : \mathscr{M} \to \mathrm{L}^\infty(\mathbf{G}_k/\mathbf{P}_k)$ a $\mathbf{G}_k$-boundary structure such that $\Theta(\mathscr{M}) \neq \mathbb{C}1$.*

Then there exists a unique proper parabolic k-subgroup $\mathbf{P} < \mathbf{Q} < \mathbf{G}$ such that $\mathrm{mult}(\Theta) \cong \mathrm{L}^\infty(\mathbf{G}_k/\mathbf{Q}_k)$ as $\mathbf{G}_k$-von Neumann algebras.

Let us explain why Theorem 3.2 implies Theorem 3.1. Let G be a higher rank simple connected real Lie group with finite center, $\mathscr{M}$ an ergodic G-von Neumann algebra and $\Theta : \mathscr{M} \to \mathrm{L}^\infty(G/P)$ a G-boundary structure such that $\Theta(\mathscr{M}) \neq \mathbb{C}1$.

Firstly, we reduce to the case where G has trivial center. Set $G_0 = G/\mathscr{Z}(G)$ and consider the canonical factor map $\pi : G \to G_0$. Observe that since $\mathscr{Z}(G) < P$, the action $\mathscr{Z}(G) \curvearrowright G/P$ is trivial. Set $P_0 = P/\mathscr{Z}(G)$ and identify $G/P \cong G_0/P_0$ as G-spaces. Set $\mathscr{M}_0 = \mathscr{M}^{\mathscr{Z}(G)}$. Then $\mathscr{M}_0$ is an ergodic G_0-von Neumann algebra and $\Theta_0 = \Theta|_{\mathscr{M}_0} : \mathscr{M}_0 \to \mathrm{L}^\infty(G_0/P_0)$ is a G_0-boundary structure. Consider the faithful normal conditional expectation $\mathrm{E} : \mathscr{M} \to \mathscr{M}_0 : x \mapsto \frac{1}{|\mathscr{Z}(G)|}\sum_{g\in\mathscr{Z}(G)} \sigma_g(x)$. Then for every $x \in \mathscr{M}$ and every $g \in G$, we have $\Theta(\sigma_g(x)) = \Theta(x)$ and so $\Theta(\mathrm{E}(x)) = \Theta(x)$. Since $\Theta(\mathscr{M}) \neq \mathbb{C}1$, we have $\Theta_0(\mathscr{M}_0) \neq \mathbb{C}1$.

Secondly, we explain how to apply Theorem 3.2 to prove Theorem 3.1. From the discussion in the previous paragraph, we may assume that G is a higher rank simple connected real Lie group with trivial center. Then there exists a simple connected algebraic $\mathbb{R}$-group $\mathbf{G}$ such that $G \cong \mathbf{G}_\mathbb{R}^0$ as Lie groups (see e.g. [Zi84, Proposition 3.1.6]). Note that $\mathbf{G}_\mathbb{R}^0 \lhd \mathbf{G}_\mathbb{R}$ is a finite index closed normal subgroup. Fix a maximal $\mathbb{R}$-split torus $\mathbf{S} < \mathbf{G}$ and a minimal parabolic $\mathbb{R}$-subgroup $\mathbf{P} < \mathbf{G}$ such that $\mathbf{S} < \mathbf{P} < \mathbf{G}$. Denote by $\mathbf{V} < \mathbf{P}$ the unipotent radical of $\mathbf{P}$, which is an $\mathbb{R}$-subgroup. Then we have $\mathbf{P} = \mathscr{Z}_\mathbf{G}(\mathbf{S}) \ltimes \mathbf{V}$. Since $\mathbf{G}_\mathbb{R} = G \cdot \mathscr{Z}_\mathbf{G}(\mathbf{S})_\mathbb{R}$ (see [Ma91, Proposition I.1.5.4]), the action $G \curvearrowright \mathbf{G}_\mathbb{R}/\mathbf{P}_\mathbb{R}$ is transitive. Then $P = G \cap \mathbf{P}_\mathbb{R}$ is a parabolic subgroup of G and we have $G/P \cong \mathbf{G}_\mathbb{R}/\mathbf{P}_\mathbb{R}$ as compact G-spaces. Consider the induced ergodic $\mathbf{G}_\mathbb{R}$-von Neumann algebra $\mathscr{M}_0 = \mathrm{Ind}_G^{\mathbf{G}_\mathbb{R}}(\mathscr{M})$. Set $\mathscr{N}_0 = \mathrm{Ind}_G^{\mathbf{G}_\mathbb{R}}(\mathrm{L}^\infty(G/P))$. Since $G/P \cong \mathbf{G}_\mathbb{R}/\mathbf{P}_\mathbb{R}$ is a $\mathbf{G}_\mathbb{R}$-space, we have $\mathscr{N}_0 \cong \mathrm{L}^\infty(\mathbf{G}_\mathbb{R}/G) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(\mathbf{G}_\mathbb{R}/\mathbf{P}_\mathbb{R})$ where $\mathbf{G}_\mathbb{R} \curvearrowright \mathbf{G}_\mathbb{R}/G \times \mathbf{G}_\mathbb{R}/\mathbf{P}_\mathbb{R}$ acts diagonally. Since $\mathbf{G}_\mathbb{R}/G$ is a finite group, we may consider the $\mathbf{G}_\mathbb{R}$-invariant probability measure $\zeta \in \mathrm{Prob}(\mathbf{G}_\mathbb{R}/G)$. Then $\Theta_0 = \zeta \otimes \Theta : \mathscr{M}_0 \to \mathrm{L}^\infty(\mathbf{G}_\mathbb{R}/\mathbf{P}_\mathbb{R})$ is a $\mathbf{G}_\mathbb{R}$-boundary structure such that $\Theta_0(\mathscr{M}_0) \neq \mathbb{C}1$. By Theorem 3.2, there exists a unique proper parabolic $\mathbb{R}$-subgroup $\mathbf{P} < \mathbf{Q} < \mathbf{G}$ such that $\mathrm{mult}(\Theta_0) \cong \mathrm{L}^\infty(\mathbf{G}_\mathbb{R}/\mathbf{Q}_\mathbb{R})$ as $\mathbf{G}_\mathbb{R}$-von Neumann algebras. Set $Q = G \cap \mathbf{Q}_\mathbb{R}$ and observe that $G/Q \cong \mathbf{G}_\mathbb{R}/\mathbf{Q}_\mathbb{R}$ as compact G-spaces. Regarding $\mathscr{M}_0 = \mathrm{L}^\infty(\mathbf{G}_\mathbb{R}, \mathscr{M})^G$, consider the well-defined G-equivariant normal unital $*$-homomorphism $\rho : \mathscr{M}_0 \to \mathscr{M} : F \mapsto F(e)$ that consists of the evaluation map at $e \in \mathbf{G}_\mathbb{R}$. Then $\rho|_{\mathrm{L}^\infty(G/Q)} : \mathrm{L}^\infty(G/Q) \to \mathscr{M}$ is a G-equivariant normal unital $*$-homomorphism such that $\Theta \circ \rho : \mathrm{L}^\infty(G/Q) \hookrightarrow \mathrm{L}^\infty(G/P)$ is the canonical inclusion by (2.1). This further implies that $\mathrm{mult}(\Theta) \cong \mathrm{L}^\infty(G/Q)$ as G-von Neumann algebras. Therefore, Theorem 3.2 implies Theorem 3.1.

3.2. **Reduction to the commutative setting.** We will be using the following notation for the rest of these lectures notes. We refer the reader to [Ma91, Chapter I] for further details. Let k be a local field of characteristic zero. Let $\mathbf{G}$ be a simple connected algebraic k-group. Fix a maximal k-split torus $\mathbf{S} < \mathbf{G}$ and a minimal parabolic k-subgroup $\mathbf{P} < \mathbf{G}$ such that $\mathbf{S} < \mathbf{P} < \mathbf{G}$. Denote by $\mathbf{V} = R_u(\mathbf{P})$ the unipotent radical of $\mathbf{P}$, which is a k-subgroup. Then we have $\mathbf{P} = \mathscr{Z}_{\mathbf{G}}(\mathbf{S}) \ltimes \mathbf{V}$. Denote by Φ^+, Φ^-, Δ the set of positive, negative and simple (positive) roots with respect to a given fixed ordering. For every subset $\theta \subset \Delta$, denote by $\mathbf{S}_\theta$, $\mathbf{P}_\theta$, $\mathbf{V}_\theta$ the corresponding k-split torus, parabolic k-subgroup and unipotent radical k-subgroup so that $\mathbf{P}_\theta = \mathscr{Z}_{\mathbf{G}}(\mathbf{S}_\theta) \ltimes \mathbf{V}_\theta$. We consider $\mathbf{R}_\theta = \mathbf{S}_\theta \ltimes \mathbf{V}_\theta$ which is the solvable radical of $\mathbf{P}_\theta$. We also denote by $\overline{\mathbf{P}}_\theta$, $\overline{\mathbf{V}}_\theta$, $\overline{\mathbf{R}}_\theta$ the corresponding opposite parabolic k-subgroup, opposite unipotent radical k-subgroup and opposite solvable radical so that $\overline{\mathbf{P}}_\theta = \mathscr{Z}_{\mathbf{G}}(\mathbf{S}_\theta) \ltimes \overline{\mathbf{V}}_\theta$ and $\overline{\mathbf{R}}_\theta = \mathbf{S}_\theta \ltimes \overline{\mathbf{V}}_\theta$. Then we have $\overline{\mathbf{V}}_\theta \cap \mathbf{P}_\theta = \{e\}$ and $\overline{\mathbf{P}}_\theta \cap \mathbf{P}_\theta = \mathscr{Z}_{\mathbf{G}}(\mathbf{S}_\theta)$. Letting $\overline{\mathbf{U}}_\theta = \mathscr{Z}_{\mathbf{G}}(\mathbf{S}_\theta) \cap \overline{\mathbf{V}} = \mathbf{P}_\theta \cap \overline{\mathbf{V}}$, we have $\overline{\mathbf{V}} = \overline{\mathbf{U}}_\theta \ltimes \overline{\mathbf{V}}_\theta$. Observe that $\mathbf{P}_\emptyset = \mathbf{P}$ and $\mathbf{P}_\Delta = \mathbf{G}$. More generally, $\mathbf{P} < \mathbf{P}_\theta < \mathbf{G}$ for $\theta \subset \Delta$ are the intermediate parabolic k-subgroups. We will simply denote by $G, S_\theta, P_\theta, V_\theta, \overline{P}_\theta, \overline{V}_\theta, \overline{U}_\theta$ the corresponding locally compact groups of k-points. By [Bo91, Proposition 20.5], we have a natural identification $G/P_\theta = (\mathbf{G}/\mathbf{P}_\theta)_k$.

The multiplication map $\overline{\mathbf{V}}_\theta \times \mathbf{P}_\theta \to \mathbf{G}$ is a k-isomorphism such that $\overline{\mathbf{V}}_\theta \cdot \mathbf{P}_\theta \subset \mathbf{G}$ is a Zariski dense open subset. At the level of k-points, the subset $\overline{V}_\theta \cdot P_\theta \subset G$ is conull. Moreover, the map $\Psi : \overline{V}_\theta \to G/P_\theta : \overline{v} \mapsto \overline{v}P_\theta$ is a measure class preserving measurable isomorphism, where $\overline{V}_\theta$ is endowed with its unique Haar measure class and G/P_θ with its unique G-invariant measure class. The map Ψ satisfies the following equivariance property: for every $s \in S_\theta$, every $\overline{v} \in \overline{V}_\theta$ and almost every $\overline{w} \in \overline{V}_\theta$, we have

$$\Psi(\overline{v}\overline{w}) = \overline{v}\Psi(\overline{w}) \quad \text{and} \quad \Psi(s\overline{w}s^{-1}) = s\Psi(\overline{w}). \tag{3.1}$$

We denote by G^+ the closed normal subgroup generated by $R_u(\mathbf{Q})_k$ where $\mathbf{Q} < \mathbf{G}$ runs through the set of all parabolic k-subgroups. Then we have $G = G^+ \cdot \mathscr{Z}_{\mathbf{G}}(\mathbf{S})_k$ and $G^+ \lhd G$ has finite index. In particular, the action $G^+ \curvearrowright G/P$ is transitive. In case $k = \mathbb{R}$, $G^+ = G^0$ coincides with the connected component of $e \in G$. In case $k = \mathbb{C}$, we have $G^+ = G$. By Theorem 2.4 (see also [BS04, Corollary 5.2]), for every closed subgroup $G^+ < H < G$ and every admissible Borel probability measure $\mu \in \mathrm{Prob}(H)$, there exists a unique μ-stationary Borel probability measure $\nu_P \in \mathrm{Prob}(G/P)$ and $(G/P, \nu_P)$ is the (H, μ)-Poisson boundary. For every intermediate parabolic k-subgroup $\mathbf{Q} = \mathbf{P}_\theta$, we denote by $\nu_Q \in \mathrm{Prob}(G/Q)$ the μ-stationary Borel probability measure that is the image of $\nu_P \in \mathrm{Prob}(G/P)$ under the factor map $p_Q : G/P \to G/Q$, where $Q = \mathbf{Q}_k$.

We illustrate the above notation in the case when $k = \mathbb{R}$, $\mathbf{G} = \mathrm{SL}_3$ and $G = \mathrm{SL}_3(\mathbb{R})$. We denote by

$$S = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mid \lambda_1, \lambda_2, \lambda_3 > 0,\ \lambda_1\lambda_2\lambda_3 = 1 \right\}$$

the Cartan subgroup. Set

$$P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $P < G$ is a minimal parabolic subgroup, V is the unipotent radical of P and we have $P = \mathscr{Z}_G(S) \ltimes V$. There are exactly two intermediate proper parabolic subgroups $P < P_1, P_2 < G$. Indeed, set

$$P_1 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $P_1 < G$ is a maximal parabolic subgroup, V_1 is its unipotent radical and we have $V = U_1 \ltimes V_1$. Define

$$S_1 = \left\{ \begin{pmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \mid \lambda > 0 \right\} \quad \text{and} \quad R_1 = S_1 \ltimes V_1.$$

Then $P_1/R_1 \cong \mathrm{PSL}_2(\mathbb{R})$. Consider the opposite subgroups $\overline{P}_1, \overline{V}_1, \overline{U}_1$. Then $P_1 \cap \overline{P}_1 = \mathscr{Z}_G(S_1)$ and $P_1 = \mathscr{Z}_G(S_1) \ltimes V_1$ is the Levi decomposition of P_1. Likewise, set

$$P_2 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $P_2 < G$ is a maximal parabolic subgroup, V_2 is its unipotent radical and we have $V = U_2 \ltimes V_2$. Define

$$S_2 = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix} \mid \lambda > 0 \right\} \quad \text{and} \quad R_2 = S_2 \ltimes V_2.$$

Then $P_2/R_2 \cong \mathrm{PSL}_2(\mathbb{R})$. Consider the opposite subgroups $\overline{P}_2, \overline{V}_2, \overline{U}_2$. Then $P_2 \cap \overline{P}_2 = \mathscr{Z}_G(S_2)$ and $P_2 = \mathscr{Z}_G(S_2) \ltimes V_2$ is the Levi decomposition of P_2.

In this subsection, we explain the first step of the proof of Theorem 3.2 that consists of reducing the problem to a commutative setting where the group action is still faithful and moreover possesses large stabilizers (see [BH19, Theorem 5.1] for the case of Lie groups and [BBH21, Theorem 3.1] for the case of algebraic groups defined over local fields of arbitrary characteristic).

Theorem 3.3. *Let k be a local field of characteristic zero. Let $\mathbf{G}$ be a simple connected algebraic k-group such that $\mathrm{rk}_k(\mathbf{G}) \geq 2$. Let $\mathscr{M}$ be a G-von Neumann algebra and $\Theta : \mathscr{M} \to \mathrm{L}^\infty(G/P)$ a faithful G-boundary structure such that $\Theta(\mathscr{M}) \neq \mathbb{C}1$. Fix a Borel probability measure $\nu_P \in \mathrm{Prob}(G/P)$ in the unique G-invariant measure class and set $\varphi = \nu_P \circ \Theta \in \mathscr{M}_\ast^+$.*

Then there exists a separable G-invariant abelian von Neumann subalgebra $\mathscr{Z} \subset \mathscr{M}$ such that $G^+ \curvearrowright \mathscr{Z}$ is faithful. Moreover, letting $(\mathscr{Z}, \varphi|_{\mathscr{Z}}) \cong \mathrm{L}^\infty(X, \nu)$, we have that $\mathrm{Stab}_G(x)$ contains the group of k-points of a nontrivial k-split torus of $\mathbf{G}$ for ν-almost every $x \in X$.

Proof. Since $\Theta(\mathscr{M}) \neq \mathbb{C}1$, there exists a separable G-invariant von Neumann subalgebra $\mathscr{M}_0 \subset \mathscr{M}$ such that $\Theta(\mathscr{M}_0) \neq \mathbb{C}1$. Upon replacing $\mathscr{M}$ by $\mathscr{M}_0$, we may assume that $\mathscr{M}$ is separable. We divide the proof into a series of claims following [BH19, Theorem 5.1] and [BBH21, Theorem 3.1].

Since $\mathrm{rk}_k(\mathbf{G}) \geq 2$, the intersection of the von Neumann subalgebras $\mathrm{L}^\infty(G/P_\theta)$ over all $\theta \subsetneq \Delta$ is equal to $\mathbb{C}1$. Since $\Theta(\mathscr{M}) \neq \mathbb{C}1$, it follows that there exists $\theta \subsetneq \Delta$ such that $\Theta(\mathscr{M}) \not\subset \mathrm{L}^\infty(G/P_\theta)$. For notational convenience, set $P_0 = P_\theta$, $S_0 = S_\theta$, $V_0 = V_\theta$, $\overline{V}_0 = \overline{V}_\theta$, $\overline{U}_0 = P_0 \cap \overline{V}$. We have $\overline{V} = \overline{U}_0 \ltimes \overline{V}_0$. Set $R_0 = S_0 \ltimes V_0$ and $\overline{R}_0 = S_0 \ltimes \overline{V}_0$.

Denote by $\mathscr{A} \subset \mathscr{M}$ the G-continuous model of $\mathscr{M}$ and recall that the G-continuous model of $\mathrm{L}^\infty(G/P)$ is equal to $\mathrm{C}(G/P)$. Since $\Theta(\mathscr{A}) \subset \mathrm{C}(G/P)$, we may regard $\Theta : \mathscr{A} \to \mathrm{C}(G/P)$ as a G-equivariant ucp map. By duality, denote by $\beta : G/P \to \mathfrak{S}(\mathscr{A}) : b \mapsto \beta_b$ the unique G-equivariant continuous map such that

$$\forall g \in G, \forall b \in G/P, \forall a \in \mathscr{A}, \quad \beta_b(a) = \Theta(a)(b).$$

Set $\psi = \beta_P \in \mathfrak{S}(\mathscr{A})^P$. Consider the GNS triple $(\pi_\psi, \mathscr{H}_\psi, \xi_\psi)$ associated with $\psi \in \mathfrak{S}(A)^P$ and set $\mathscr{N} = \pi_\psi(\mathscr{A})'' \subset \mathrm{B}(\mathscr{H}_\psi)$. We still denote by $\psi = \langle \cdot \xi_\psi, \xi_\psi \rangle$ the normal state on $\mathscr{N}$ and we observe that for every $a \in \mathscr{A}$, we have $\psi(\pi_\psi(a)) = \psi(a)$. In the commutative case, namely when $\mathscr{M}$ is abelian, $\mathscr{A}$ is abelian and so $\psi \in \mathscr{N}_\ast$ is faithful on $\mathscr{N}$. In the noncommutative case, namely when $\mathscr{M}$ is noncommutative, the normal state $\psi \in \mathscr{N}_\ast$ need not be faithful on $\mathscr{N}$. The ψ-preserving continuous action $P \curvearrowright \mathscr{A}$ extends to a ψ-preserving continuous action $\sigma^{\mathscr{N}} : P \curvearrowright \mathscr{N}$.

In the **first part of the proof** (see Claims 3.4, 3.5, 3.6, 3.7 below), following Nevo–Zimmer's approach [NZ00], we construct a nontrivial G-invariant von Neumann subalgebra $\mathscr{M}_0 \subset \mathscr{M}$ such that $\mathscr{M}_0 \subset \mathrm{Ind}_P^G(\mathscr{N}_0)$, where $\mathscr{N}_0 \subset q\mathscr{N}q$ is a P-invariant von Neumann subalgebra and $q \in \mathscr{N}^P$, and for which the action $R_0 \curvearrowright \mathscr{N}_0$ is trivial.

Choose a Borel section $\tau : G/P \to G$ and consider the Borel 1-cocycle $c : G \times G/P \to P : (g, b) \mapsto \tau(gb)^{-1} g \tau(b)$. The induced action $G \curvearrowright \mathrm{Ind}_P^G(\mathscr{N})$ is given by the formula

$$\forall g \in G, \forall F \in \mathrm{Ind}_P^G(\mathscr{N}), \quad \sigma_g(F)(b) = \sigma^{\mathscr{N}}_{c(g, g^{-1}b)}(F(g^{-1}b)).$$

Consider the map $\iota : \mathcal{A} \to \mathrm{Ind}_P^G(\mathscr{N})$ defined by the formula

$$\forall a \in \mathcal{A}, \forall b \in G/P, \quad \iota(a)(b) = \pi_\psi(\sigma_{\tau(b)}^{-1}(a)).$$

Since $\psi \in \mathscr{N}_*$ is P-invariant, we may consider the G-equivariant normal ucp map $\widehat{\Theta} = \mathrm{id}_{G/P} \otimes \psi : \mathrm{Ind}_P^G(\mathscr{N}) \to \mathrm{L}^\infty(G/P)$.

Claim 3.4. We may extend $\iota : \mathscr{M} \to \mathrm{Ind}_P^G(\mathscr{N})$ to a G-equivariant normal unital $*$-embedding. Moreover, we have $\widehat{\Theta} \circ \iota = \Theta$.

Proof of Claim 3.4. Firstly, observe that for every $a \in \mathcal{A}$, every $g \in G$ and every $b \in G/P$, we have

$$\begin{aligned}\iota(\sigma_g(a))(b) &= \pi_\psi(\sigma_{\tau(b)}^{-1}(\sigma_g(a))) \\ &= \pi_\psi(\sigma_{\tau(b)^{-1} g \tau(g^{-1}b)\tau(g^{-1}b)^{-1}}(a)) \\ &= \sigma^{\mathscr{N}}_{c(g,g^{-1}b)}(\pi_\psi(\sigma_{\tau(g^{-1}b)}^{-1}(a))) \\ &= \sigma_g(\iota(a))(b)\end{aligned}$$

and so $\iota(\sigma_g(a)) = \sigma_g(\iota(a))$. Secondly, observe that for every $a \in \mathcal{A}$ and every $b \in G/P$, we have

$$\widehat{\Theta}(\iota(a))(b) = \psi(\sigma_{\tau(b)}^{-1}(a)) = \beta_{\tau(b)P}(a) = \beta_b(a) = \Theta(a)(b)$$

and so $\widehat{\Theta} \circ \iota = \Theta$. Thus, once we proved that $\iota : \mathscr{M} \to \mathrm{Ind}_P^G(\mathscr{N})$ extends to a normal unital $*$-embedding, we will necessarily have that ι is G-equivariant and $\widehat{\Theta} \circ \iota = \Theta$.

Set $\mathscr{H} = \mathrm{L}^2(G/P, \nu_P) \otimes \mathscr{H}_\psi$ and $\xi = \mathbf{1}_{G/P} \otimes \xi_\psi \in \mathscr{H}$. Denote by $p \in \iota(\mathscr{A})' \cap \mathrm{B}(\mathscr{H})$ the orthogonal projection onto the closed linear span $\mathscr{K} = \overline{\iota(\mathscr{A})\xi}$. We identify $\mathrm{B}(\mathscr{K}) = p\,\mathrm{B}(\mathscr{H})p$. Observe that ξ is a $\iota(\mathscr{A})$-cyclic vector in $\mathscr{K}$ that implements the state φ on $\mathscr{A}$. Thus, by uniqueness of the GNS representation, the unital $*$-representation $\mathscr{A} \to \mathrm{B}(\mathscr{K}) : a \mapsto \iota(a)p$ is unitarily conjugate to $\pi_\varphi = \mathrm{id}$. In particular, it indeed extends to a normal unital $*$-isomorphism $\mathscr{M} \to \iota(\mathscr{A})''p : a \mapsto \iota(a)p$. We are left to check that the normal unital $*$-homomorphism $\iota(\mathscr{A})'' \to \iota(\mathscr{A})''p : f \mapsto fp$ is injective. Let $f \in \iota(\mathscr{A})''$ be such that $fp = 0$. For every $a \in \mathscr{A}$, we have $f\,\iota(a)\xi = 0$. Regarding $f \in \mathrm{L}^\infty(G/P, \mathscr{N})$, for every $a \in \mathscr{A}$ and almost every $b \in G/P$, we have $f(b)\,\pi_\psi(\sigma_{\tau(b)}^{-1}(a))\xi_\psi = 0$. Since $\mathcal{A}$ is separable, this implies that for almost every $b \in G/P$ and every $a \in \mathcal{A}$, we have $f(b)\,\pi_\psi(\sigma_{\tau(b)}^{-1}(a))\xi_\psi = 0$. Since ξ_ψ is $\pi_\psi(\mathcal{A})$-cyclic, we conclude that $f(b) = 0$ for almost every $b \in G/P$. This finally shows that $f = 0$. $\square$

From now on, we will use the letter σ to denote any of the actions involved.

Claim 3.5. We have $\Theta(\mathscr{M}^{\overline{V}_0}) \neq \mathbb{C}1$.

Proof of Claim 3.5. Regard $\mathrm{L}^\infty(G/P) = \mathrm{L}^\infty(\overline{V}) = \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(\overline{U}_0) = \mathrm{L}^\infty(\overline{V}_0, \mathscr{Q}_0)$, where $\mathscr{Q}_0 = \mathrm{L}^\infty(\overline{U}_0) = \mathrm{L}^\infty(P_0/P)$. We identify $\mathrm{L}^\infty(G/P_0) =$

$L^\infty(\overline{V}_0) \otimes \mathbb{C}1 \subset L^\infty(\overline{V}_0, \mathscr{Q}_0)$. Recall that the actions $S_0 \curvearrowright L^\infty(\overline{V}_0, \mathscr{Q}_0)$ and $\overline{V}_0 \curvearrowright L^\infty(\overline{V}_0, \mathscr{Q}_0)$ are given by

$$\forall s \in S_0, \forall \overline{v} \in \overline{V}_0, \forall f \in L^\infty(\overline{V}_0, \mathscr{Q}_0),$$
$$\sigma_s(f)(\overline{w}) = f(s^{-1}\overline{w}s) \quad \text{and} \quad \sigma_{\overline{v}}(f)(\overline{w}) = f(\overline{v}^{-1}\overline{w}).$$

Denote by $\mathscr{B} \subset \mathscr{M}$ the $\overline{V}_0$-continuous model of $\mathscr{M}$. Then we have $\Theta(\mathscr{B}) \subset C_b(\overline{V}_0, \mathscr{Q}_0)$. Since $\Theta(\mathscr{M}) \not\subset L^\infty(G/P_0)$, there exists $b \in \mathscr{B}$ such that $y = \Theta(b)(e) \in \mathscr{Q}_0 \setminus \mathbb{C}1$. By the contraction property, we may choose $s \in S_0$ such that $\lim_n s^{-n}\overline{v}s^n = e$ for every $\overline{v} \in \overline{V}_0$. Choose a nonprincipal ultrafilter $\mathcal{U} \in \beta(\mathbb{N}) \setminus \mathbb{N}$ and define the ucp map $E_s : \mathscr{M} \to \mathscr{M}^s : x \mapsto \lim_{\mathcal{U}} \frac{1}{n}\sum_{j=1}^n \sigma_{s^j}(x)$ where the limit is taken with respect to the weak-$*$ topology. We prove the following fact that implies Claim 3.5.

Fact. We have $E_s(b) \in \mathscr{M}^{\overline{V}_0} = \mathscr{B}^{\overline{V}_0}$ and $\Theta(E_s(b)) = \mathbf{1}_{\overline{V}_0} \otimes y \notin \mathbb{C}1$.

For every $n \geq 1$, set $b_n = \frac{1}{n}\sum_{j=1}^n \sigma_{s^j}(b) \in \mathscr{B}$. Then we have

$$\forall n \geq 1, \forall \overline{v} \in \overline{V}_0, \quad \|\sigma_{\overline{v}}(b_n) - b_n\| \leq \frac{1}{n}\sum_{j=1}^n \|\sigma_{s^{-j}\overline{v}s^j}(b) - b\|.$$

Thus, we have $\lim_n \|\sigma_{\overline{v}}(b_n) - b_n\| = 0$. This further implies that $\sigma_{\overline{v}}(E_s(b)) = E_s(b)$ for every $\overline{v} \in \overline{V}_0$ and so $E_s(b) \in \mathscr{M}^{\overline{V}_0} = \mathscr{B}^{\overline{V}_0}$. Since the action $S_0 \curvearrowright \mathscr{Q}_0$ is trivial, we have $\Theta \circ \sigma_s = (\sigma_s \otimes \mathrm{id}) \circ \Theta$. Set $f = \Theta(b) \in C_b(\overline{V}_0, \mathscr{Q}_0)$. By the contraction property, $(\sigma_{s^n} \otimes \mathrm{id})(f) \to \mathbf{1}_{\overline{V}_0} \otimes f(e)$ pointwise on $\overline{V}_0$. Lebesgue's dominated convergence theorem further implies that $(\sigma_{s^n} \otimes \mathrm{id})(f) \to \mathbf{1}_{\overline{V}_0} \otimes f(e)$ with respect to the weak-$*$ topology in $L^\infty(\overline{V}_0, \mathscr{Q}_0)$. Since $\Theta : \mathscr{M} \to L^\infty(\overline{V}_0, \mathscr{Q}_0)$ is normal, we have

$$\Theta(E_s(b)) = \lim_{\mathcal{U}} \frac{1}{n}\sum_{j=1}^n (\sigma_{s^j} \otimes \mathrm{id})(\Theta(b)) = \mathbf{1}_{\overline{V}_0} \otimes f(e) = \mathbf{1}_{\overline{V}_0} \otimes y.$$

This finishes the proof of Claim 3.5. □

Claim 3.6. We have $\Theta(\mathscr{M}^{\overline{R}_0}) \neq \mathbb{C}1$.

Proof of Claim 3.6. Consider the nonempty convex set

$$\mathscr{C} = \left\{\Phi : \mathscr{M}^{\overline{V}_0} \to \mathscr{M}^{\overline{V}_0} \text{ ucp map} \mid \Theta \circ \Phi = \Theta|_{\mathscr{M}^{\overline{V}_0}}\right\}.$$

Observe that $\mathscr{C}$ is compact for the pointwise weak-$*$ topology on $\mathscr{M}^{\overline{V}_0}$. Since the action $S_0 \curvearrowright \mathscr{Q}_0$ is trivial and since $\mathscr{Q}_0 = L^\infty(G/P)^{\overline{V}_0}$, we may define the affine action $S_0 \curvearrowright \mathscr{C}$ by the formula $s \cdot \Phi = \sigma_s \circ \Phi$ for every $s \in S_0$ and every $\Phi \in \mathscr{C}$. Since S_0 is abelian and so amenable (as a discrete group), there exists $\Phi \in \mathscr{C}$ such that $\sigma_s \circ \Phi = \Phi$ for every $s \in S_0$. Then we have $\Phi(\mathscr{M}^{\overline{V}_0}) \subset \mathscr{M}^{\overline{R}_0}$. Using Claim 3.5, we have $\mathbb{C}1 \neq \Theta(\mathscr{M}^{\overline{V}_0}) = \Theta(\Phi(\mathscr{M}^{\overline{V}_0})) \subset \Theta(\mathscr{M}^{\overline{R}_0})$. This finishes the proof of Claim 3.6. □

By Claim 3.4, we may regard $\mathscr{M} \subset \operatorname{Ind}_P^G(\mathscr{N})$ as a G-invariant von Neumann subalgebra such that $\Theta = \widehat{\Theta}|_{\mathscr{M}}$. We show that under a well-chosen nonzero G-invariant projection, the fixed point von Neumann subalgebra $\operatorname{Ind}_P^G(\mathscr{N})^{\overline{R}_0}$ is included in an induced von Neumann algebra. Denote by $q \in (\mathscr{N}^{R_0})' \cap \mathscr{N}$ the support projection of the normal state $\psi|_{(\mathscr{N}^{R_0})' \cap \mathscr{N}} \in ((\mathscr{N}^{R_0})' \cap \mathscr{N})_*$. Since $R_0 \lhd P_0$ is a normal subgroup and since $R_0 < P$, it follows that $(\mathscr{N}^{R_0})' \cap \mathscr{N} \subset \mathscr{N}$ is globally P-invariant. This implies that $q \in (\mathscr{N}^{R_0})' \cap \mathscr{N}^P = \mathscr{Z}(\mathscr{N}^{R_0}) \cap \mathscr{N}^P$. Define $p = \mathbf{1}_{G/P} \otimes q \in \operatorname{Ind}_P^G(\mathscr{N})$ and observe that $p \in \operatorname{Ind}_P^G(\mathscr{N})^G$.

Claim 3.7. We have $p \in \mathscr{Z}(\operatorname{Ind}_P^G(\mathscr{N})^{\overline{R}_0}) \cap \operatorname{Ind}_P^G(\mathscr{N})^G$, $\widehat{\Theta}(p) = 1$ and $\operatorname{Ind}_P^G(\mathscr{N})^{\overline{R}_0} p \subset \operatorname{Ind}_P^G(\mathscr{N}^{R_0} q)$.

Proof. In order to prove the claim, we prove the following useful fact.

Fact. We have $q \in (\mathscr{N}^{S_0})' \cap \mathscr{N}$ and $\mathscr{N}^{S_0} q = \mathscr{N}^{R_0} q$.

Regard $\mathscr{N} \subset \mathrm{B}(\mathscr{H}_\psi)$ via the GNS representation. Then $q : \mathscr{H}_\psi \to [(\mathscr{N}' \vee \mathscr{N}^{R_0})\xi_\psi]$ is the orthogonal projection onto the closed linear subspace generated by $(\mathscr{N}' \vee \mathscr{N}^{R_0})\xi_\psi$. Let $x \in \mathscr{N}^{S_0}$ and $v \in V_0$. Choose a sequence $(s_n)_{n\in\mathbb{N}}$ in S_0 such that $\lim_n s_n v s_n^{-1} = e$. Then for every $y \in \mathscr{N}'$ and every $z \in \mathscr{N}^{R_0}$, we have

$$\begin{aligned}\|(\sigma_v(x) - x)yz\xi_\psi\| &= \|y(\sigma_v(x) - x)z\xi_\psi\| \\ &\leq \|y\|_\infty \|(\sigma_{s_n v s_n^{-1}}(xz) - xz)\xi_\psi\| \quad \to \quad 0.\end{aligned}$$

Thus, we have $\sigma_v(xq) = \sigma_v(x)q = xq$. This implies that $xq \in \mathscr{N}^{R_0}$. Since $q \in \mathscr{Z}(\mathscr{N}^{R_0})$, we have $xq = qxq$. Applying this equality to $x^* \in \mathscr{N}^{S_0}$, we also have $x^*q = qx^*q$ and so $qx = qxq$. This further implies that $qx = qxq = xq \in \mathscr{N}^{R_0}$. Therefore, we have $q \in (\mathscr{N}^{S_0})' \cap \mathscr{N}$ and $\mathscr{N}^{S_0} q = \mathscr{N}^{R_0} q$. This finishes the proof of the fact.

Observe that $\operatorname{Ind}_P^G(\mathscr{N}) = \operatorname{Ind}_{P_0}^G(\operatorname{Ind}_P^{P_0}(\mathscr{N}))$. Since $\mathrm{L}^\infty(\overline{V}_0) = \mathrm{L}^\infty(G/P_0)$, we have

$$\operatorname{Ind}_P^G(\mathscr{N})^{\overline{R}_0} = \mathrm{L}^\infty(\overline{V}_0, \operatorname{Ind}_P^{P_0}(\mathscr{N}))^{\overline{R}_0} = \mathbb{C}\mathbf{1}_{\overline{V}_0} \otimes \operatorname{Ind}_P^{P_0}(\mathscr{N})^{S_0}.$$

Moreover, since $R_0 \lhd P_0$ is a normal subgroup and since $R_0 < P$, the action $R_0 \curvearrowright \mathrm{L}^\infty(P_0/P)$ is trivial and $\mathscr{N}^{R_0} \subset \mathscr{N}$ is globally P-invariant, which further implies that $\operatorname{Ind}_P^{P_0}(\mathscr{N})^{R_0} = \operatorname{Ind}_P^{P_0}(\mathscr{N}^{R_0})$. Moreover, since $\mathscr{N}^{S_0} q = \mathscr{N}^{R_0} q \subset q\mathscr{N} q$ is P-invariant, we have

$$\begin{aligned}\operatorname{Ind}_P^{P_0}(\mathscr{N})^{S_0}(\mathbf{1}_{P_0/P} \otimes q) &= \operatorname{Ind}_P^{P_0}(\mathscr{N}^{S_0} q) \\ &= \operatorname{Ind}_P^{P_0}(\mathscr{N}^{R_0} q) \\ &= \operatorname{Ind}_P^{P_0}(\mathscr{N})^{R_0}(\mathbf{1}_{P_0/P} \otimes q)\end{aligned}$$

Since $p = \mathbf{1}_{\overline{V}_0} \otimes \mathbf{1}_{\overline{U}_0} \otimes q$, a combination of the above equality and the fact implies that $p \in \mathscr{Z}(\operatorname{Ind}_P^G(\mathscr{N})^{\overline{R}_0}) \cap \operatorname{Ind}_P^G(\mathscr{N})^G$ and $\operatorname{Ind}_P^G(\mathscr{N})^{\overline{R}_0} p \subset$

$\mathrm{Ind}_P^G(\mathscr{N}^{R_0}q)$. Moreover, since $\psi(q) = 1$, we clearly have $\Theta(p) = 1$. This finishes the proof of Claim 3.7. □

Consider the G-invariant von Neumann subalgebra $\mathscr{M}_0 \subset \mathscr{M}$ generated by $\mathscr{M}^{\overline{R}_0}$. Since $p \in \mathscr{Z}(\mathrm{Ind}_P^G(\mathscr{N})^{\overline{R}_0}) \cap \mathrm{Ind}_P^G(\mathscr{N})^G$, we have that p commutes with $\mathscr{M}_0$. Denote by $z \in \mathscr{Z}(\mathscr{M}_0)$ the unique central projection such that the map $\mathscr{M}_0 z \to \mathscr{M}_0 p$ is a normal unital $*$-isomorphism. Then we have $p = zp$ and so $1 = \widehat{\Theta}(p) = \widehat{\Theta}(z)\widehat{\Theta}(p) = \Theta(z)$. Since Θ is faithful on $\mathscr{M}$, we have $z = 1$. With a slight abuse of notation, we identify $\mathscr{M}_0 = \mathscr{M}_0 p$ and we regard $\mathscr{M}_0 \subset \mathrm{Ind}_P^G(\mathscr{N}_0)$ where $\mathscr{N}_0 = \mathscr{N}^{R_0}q$.

In the **second part of the proof** (see Claims 3.8, 3.9 below), we show that $\mathscr{Z} = \mathscr{Z}(\mathscr{M}_0)$ satisfies the conclusion of Theorem 3.3. For this, we exploit the ergodicity phenomenon arising from the conjugation action in noncommutative von Neumann algebras. This second part is conceptually new compared to [NZ00].

Claim 3.8. The action $G^+ \curvearrowright \mathscr{Z}$ is faithful.

Proof of Claim 3.8. Denote by $\mathscr{A}_0 \subset \mathscr{M}_0$ the G-continuous model of $\mathscr{M}_0$. In particular, we have $\mathscr{A}_0 \subset \mathrm{C}_b(\overline{V}_0, \mathrm{Ind}_P^{P_0}(\mathscr{N}_0))$. Then denote by $\mathscr{P}_0 \subset \mathrm{Ind}_P^{P_0}(\mathscr{N}_0)$ the von Neumann subalgebra generated by all the values of the functions $f \in \mathscr{A}_0 \subset \mathrm{C}_b(\overline{V}_0, \mathrm{Ind}_P^{P_0}(\mathscr{N}_0))$. By construction, we have $\mathscr{A}_0 \subset \mathrm{L}^\infty(\overline{V}_0, \mathscr{P}_0)$ and so $\mathscr{M}_0 \subset \mathrm{L}^\infty(\overline{V}_0, \mathscr{P}_0)$.

Firstly, we prove that $\mathbb{C}\mathbf{1}_{\overline{V}_0} \otimes \mathscr{P}_0 \subset \mathscr{M}_0$. Indeed, let $f \in \mathscr{A}_0$ and $\overline{v} \in \overline{V}_0$. It suffices to show that $f(\overline{v}) \in \mathscr{M}_0$. Upon replacing $f \in \mathscr{A}_0$ by $\sigma_{\overline{v}^{-1}}(f) \in \mathscr{A}_0$, it suffices to show that $\mathbf{1}_{\overline{V}_0} \otimes f(e) \in \mathscr{M}_0$. Choose a sequence $(s_n)_{n \in \mathbb{N}}$ in S_0 such that for every $\overline{v} \in \overline{V}_0$, we have $\lim_n s_n^{-1}\overline{v}s_n = e$. Then we have $\sigma_{s_n}(f) \to \mathbf{1}_{\overline{V}_0} \otimes f(e)$ pointwise on $\overline{V}_0$. Lebesgue's dominated convergence theorem further implies that $\sigma_{s_n}(f) \to \mathbf{1}_{\overline{V}_0} \otimes f(e)$ with respect to the weak-$*$ topology in $\mathrm{L}^\infty(\overline{V}_0, \mathscr{P}_0)$. Since $\sigma_{s_n}(f) \in \mathscr{M}_0$ for every $n \in \mathbb{N}$, we have $\mathbf{1}_{\overline{V}_0} \otimes f(e) \in \mathscr{M}_0$. Then we have $\mathbb{C}\mathbf{1}_{\overline{V}_0} \otimes \mathscr{P}_0 \subset \mathscr{M}_0 \subset \mathrm{L}^\infty(\overline{V}_0, \mathscr{P}_0)$ which further implies that

$$\mathbb{C}\mathbf{1}_{\overline{V}_0} \otimes \mathscr{Z}(\mathscr{P}_0) \subset \mathscr{Z} \subset \mathrm{L}^\infty(\overline{V}_0) \mathbin{\overline{\otimes}} \mathscr{Z}(\mathscr{P}_0). \tag{3.2}$$

Secondly, we prove that $\mathbb{C}\mathbf{1}_{\overline{V}_0} \otimes \mathscr{P}_0 \neq \mathscr{M}_0$. Indeed, by contradiction, assume that $\mathbb{C}\mathbf{1}_{\overline{V}_0} \otimes \mathscr{P}_0 = \mathscr{M}_0$. Then the action $\overline{V}_0 \curvearrowright \mathscr{M}_0$ is trivial. By Tits' simplicity theorem (see e.g. [Ma91, Theorem I.1.5.6]), the action $G^+ \curvearrowright \mathscr{M}_0$ is trivial. Since the action $G^+ \curvearrowright G/P$ is ergodic, this further implies that $\Theta(\mathscr{M}_0) = \mathbb{C}1$. This however contradicts Claim 3.6. Therefore, we have $\mathbb{C}\mathbf{1}_{\overline{V}_0} \otimes \mathscr{P}_0 \neq \mathscr{M}_0$.

Thirdly, we prove that the action $\overline{V}_0 \curvearrowright \mathscr{Z}$ is nontrivial. By Tits' simplicity theorem (see e.g. [Ma91, Theorem I.1.5.6]), this will imply that the action $G^+ \curvearrowright \mathscr{Z}$ is faithful. Indeed, by contradiction, assume that the action $\overline{V}_0 \curvearrowright \mathscr{Z}$ is trivial. Then we necessarily have $\mathbb{C}\mathbf{1}_{\overline{V}_0} \otimes \mathscr{Z}(\mathscr{P}_0) = \mathscr{Z}$. By [GK95, Theorem B], there exists a normal conditional expectation Φ :

$\mathscr{P}_0 \to \mathscr{Z}(\mathscr{P}_0)$ that is *proper* in the following sense

$$\forall x \in \mathscr{P}_0, \quad \Phi(x) \in \overline{\mathrm{conv}\,\{uxu^* \mid u \in \mathscr{U}(\mathscr{P}_0)\}}^w.$$

Then the normal conditional expectation $\mathrm{id} \otimes \Phi : \mathrm{L}^\infty(\overline{V}_0) \overline{\otimes} \mathscr{P}_0 \to \mathrm{L}^\infty(\overline{V}_0) \overline{\otimes} \mathscr{Z}(\mathscr{P}_0)$ satisfies $(\mathrm{id} \otimes \Phi)(\mathscr{M}_0) \subset \mathscr{M}_0 \cap (\mathrm{L}^\infty(\overline{V}_0) \overline{\otimes} \mathscr{Z}(\mathscr{P}_0)) = \mathscr{Z}$. Choose a faithful normal state $\rho \in \mathscr{Z}(\mathscr{P}_0)_*$ and set $\phi = \rho \circ \Phi \in (\mathscr{P}_0)_*$. Then $\phi|_{\mathscr{Z}(\mathscr{P}_0)} = \rho$ is faithful on $\mathscr{Z}(\mathscr{P}_0)$. Moreover, we have $(\mathrm{id} \otimes \phi)(\mathscr{M}_0) = \mathbb{C}1$. We need the following fact that follows from Hahn–Banach theorem.

Fact. The set $\mathscr{S} = \{a\phi b \mid a, b \in \mathscr{P}_0\}$ is $\|\cdot\|$-dense in $(\mathscr{P}_0)_*$.

Indeed, denote by $\mathscr{I} = \{x \in \mathscr{P}_0 \mid \forall \omega \in \mathscr{S},\ \omega(x) = 0\}$ the annihilator of $\mathscr{S}$ in $\mathscr{P}_0$. Then $\mathscr{I} \subset \mathscr{P}_0$ is a weak-$*$ closed two-sided ideal. Therefore, there exists $z \in \mathscr{Z}(\mathscr{P}_0)$ such that $\mathscr{I} = \mathscr{P}_0 z$. Since $\phi(z) = 0$ and $\phi|_{\mathscr{Z}(\mathscr{P}_0)}$ is faithful, we have $z = 0$. Then Hahn–Banach theorem implies that $\mathscr{S}$ is $\|\cdot\|$-dense in $(\mathscr{P}_0)_*$.

Since $\mathbb{C}\mathbf{1}_{\overline{V}_0} \otimes \mathscr{P}_0 \subset \mathscr{M}_0$, the fact implies that $(\mathrm{id} \otimes \omega)(\mathscr{M}_0) = \mathbb{C}1$ for every $\omega \in (\mathscr{P}_0)_*$. Then [GK95, Theorem C] and the Slice Mapping Theorem (see [KR92, Exercise 12.4.36(v)]) imply that $\mathscr{M}_0 = \mathbb{C}\mathbf{1}_{\overline{V}_0} \otimes \mathscr{P}_0$. This however contradicts the third paragraph of the proof. Therefore, the action $G^+ \curvearrowright \mathscr{Z}$ is faithful. This finishes the proof of Claim 3.8. □

Define $\mathscr{Z}_0 = \mathrm{Ind}_P^{P_0}(\mathscr{Z}(\mathscr{N}_0))$ and observe that the action $R_0 \curvearrowright \mathscr{Z}_0$ is trivial. Moreover, we have $\mathscr{Z} \subset \mathrm{Ind}_{P_0}^G(\mathscr{Z}_0)$. Write $(\mathscr{Z}, \varphi|_{\mathscr{Z}}) \cong \mathrm{L}^\infty(X, \nu)$ and $\mathscr{Z}_0 = \mathrm{L}^\infty(Y, \eta)$.

Claim 3.9. For ν-almost every $x \in X$, the closed subgroup $\mathrm{Stab}_G(x) < G$ contains the conjugate of the group of k-points of a nontrivial k-split torus of $\mathbf{G}$.

Proof. There exists a G-equivariant measurable map $\pi : \mathrm{Ind}_{P_0}^G(Y, \eta) \to (X, \nu)$ such that $\pi_*(\nu_{G/P_0} \otimes \eta) \sim \nu$. Choose a Borel section $\tau : G/P_0 \to G$ and consider the Borel 1-cocycle $c : G \times G/P_0 \to P_0 : (g, b) \mapsto \tau(gb)^{-1} g \tau(b)$. Then the induced Borel action $G \curvearrowright \mathrm{Ind}_{P_0}^G(Y, \eta)$ is defined by

$$\forall g \in G, \forall b \in G/P_0, \forall y \in Y, \quad g \cdot (b, y) = (gb, c(g, b)y).$$

Observe that for every $r \in R_0$, every $b \in G/P_0$ and every $y \in Y$, letting $g = \tau(b) r \tau(b)^{-1}$, we have

$$\begin{aligned} g \cdot (b, y) &= (gb, c(g, b)y) \\ &= (\tau(b) r \tau(b)^{-1} b, \tau(\tau(b) r \tau(b)^{-1} b)^{-1} \tau(b) r \tau(b)^{-1} \tau(b) y) \\ &= (b, y), \end{aligned}$$

which implies that $\tau(b) R_0 \tau(b)^{-1} < \mathrm{Stab}_G(b, y)$. Thus, by G-equivariance, for ν-almost every $x \in X$, the closed subgroup $\mathrm{Stab}_G(x) < G$ contains (a conjugate of) the group of k-points of a nontrivial k-split torus of $\mathbf{G}$. This finishes the proof of Claim 3.9. □

This finishes the proof of Theorem 3.3. □

3.3. **The Nevo–Zimmer structure theorem for stationary ergodic actions.** We prove Nevo–Zimmer's structure theorem for faithful stationary ergodic actions with large stabilizers [NZ00, Proposition 3.1] (see [BBH21, Theorem 4.1] for the case of algebraic groups defined over local fields of arbitrary characteristic).

Theorem 3.10 (Nevo–Zimmer [NZ00])**.** *Let k be a local field of characteristic zero. Let $\mathbf{G}$ be a simple connected algebraic k-group such that $\mathrm{rk}_k(\mathbf{G}) \geq 1$. Let $\mu \in \mathrm{Prob}(G)$ be an admissible Borel probability measure and (X, ν) an ergodic (G, μ)-space such that $G^+ \curvearrowright (X, \nu)$ is faithful. Assume that for ν-almost every $x \in X$, $\mathrm{Stab}_G(x)$ contains the group of k-points of a nontrivial k-split torus of $\mathbf{G}$.*

Then there exist a proper parabolic k-subgroup $\mathbf{P} < \mathbf{Q} < \mathbf{G}$ and a G-equivariant measurable factor map $(X, \nu) \to (G/Q, \nu_Q)$.

Proof. The proof relies in a crucial way on the theory of algebraic groups. We may regard G as a k-analytic Lie group and it is known that $G^+ \lhd G$ is a finite index open normal subgroup. We have $\mathrm{Lie}(G) \cong \mathrm{Lie}(\mathbf{G})_k$ as k-Lie algebras. Consider the faithful irreducible adjoint k-representation $\mathrm{Ad} : \mathbf{G} \to \mathrm{GL}(\mathrm{Lie}(\mathbf{G}))$. Then we have $\mathrm{Ad}(G) \subset \mathrm{GL}(\mathrm{Lie}(G))$. Denote by $\mathrm{Gr}(\mathrm{Lie}(G))$ (resp. $\mathrm{Gr}(\mathrm{Lie}(\mathbf{G}))$) the k-analytic Grassmannian G-manifold (resp. algebraic k-$\mathbf{G}$-variety) of all subspaces of $\mathrm{Lie}(G)$ (resp. $\mathrm{Lie}(\mathbf{G})$). We naturally have $\mathrm{Gr}(\mathrm{Lie}(G)) \cong \mathrm{Gr}(\mathrm{Lie}(\mathbf{G}))_k$ as k-analytic manifolds.

We need to introduce some further notation. Denote by $\mathrm{Sub}(G)$ the space of all closed subgroups of G. Endowed with the Chabauty topology, $\mathrm{Sub}(G)$ is a compact metrizable space and the conjugation action $G \curvearrowright \mathrm{Sub}(G)$ is continuous. By Cartan's theorem, every element $H \in \mathrm{Sub}(G)$ is a k-analytic Lie subgroup of G (see the corollary to Theorem 1 in [Se92, Part II, Chapter V, §9]). For ν-almost every $x \in X$, we have $\mathrm{Stab}_G(x) \in \mathrm{Sub}(G)$ and the stabilizer map $X \to \mathrm{Sub}(G) : x \mapsto \mathrm{Stab}_G(x)$ is G-equivariant and measurable (see [AM66, Chapter II]). The Lie algebra map $\mathrm{Sub}(G) \to \mathrm{Gr}(\mathrm{Lie}(G)) : H \mapsto \mathrm{Lie}(H)$ is G-equivariant and Borel (see [NZ00, Section 3]). Therefore, the *Gauss map* $\psi : X \to \mathrm{Gr}(\mathrm{Lie}(G)) : x \mapsto \mathrm{Lie}(\mathrm{Stab}_G(x))$ is G-equivariant and measurable.

By assumption, for ν-almost every $x \in X$, we have $0 < \dim(\psi(x))$. We claim that for ν-almost every $x \in X$, we have $\dim(\psi(x)) < \dim(\mathrm{Lie}(G))$. Indeed, otherwise by ergodicity, we have $\dim(\mathrm{Lie}(\mathrm{Stab}_G(x))) = \dim(\psi(x)) = \dim(\mathrm{Lie}(G))$ for ν-almost every $x \in X$. Then for ν-almost every $x \in X$, the subgroup $\mathrm{Stab}_G(x) < G$ is open and noncompact by assumption. Since $G^+ \lhd G$ is a finite index open subgroup with the Howe–Moore property (see [HM77, Theorem 5.1]), it follows that $G^+ \cap \mathrm{Stab}_G(x) = G^+$ and so $G^+ < \mathrm{Stab}_G(x)$ for ν-almost every $x \in X$. This however contradicts the assumption that the action $G^+ \curvearrowright (X, \nu)$ is faithful. Therefore, $0 < \dim(\psi(x)) < \dim(\mathrm{Lie}(G))$ for ν-almost every $x \in X$.

Since $\mathrm{Gr}(\mathrm{Lie}(\mathbf{G}))$ is an algebraic k-$\mathbf{G}$-variety, the Borel action $G \curvearrowright \mathrm{Gr}(\mathrm{Lie}(\mathbf{G}))_k$ is tame and so the quotient Borel space $G \backslash \mathrm{Gr}(\mathrm{Lie}(\mathbf{G}))_k$ is a standard Borel space (see [Zi84, Theorem 2.1.14, Proposition 3.1.3]). Since the action $G \curvearrowright (X, \nu)$ is ergodic, the invariant map $X \to G \backslash \operatorname{Gr}(\operatorname{Lie}(\mathbf{G}))_k$ is ν-almost everywhere constant and so there exists $w \in \operatorname{Gr}(\operatorname{Lie}(G)) = \operatorname{Gr}(\operatorname{Lie}(\mathbf{G}))_k$ such that $\operatorname{Lie}(\operatorname{Stab}_G(x)) = \psi(x) \in Gw$ for ν-almost every $x \in X$. Then $\mathbf{H} = \operatorname{Stab}_{\mathbf{G}}(w) < \mathbf{G}$ is a k-subgroup and we have a k-isomorphism $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{G}w \subset \operatorname{Gr}(\operatorname{Lie}(\mathbf{G}))$ such that $\beta|_{G/H} : G/H \to Gw$ is a homeomorphism. Here we regard $G/H \subset (\mathbf{G}/\mathbf{H})_k$ where $H = \mathbf{H}_k$. Thus, upon replacing ψ by $(\beta|_{G/H})^{-1} \circ \psi$, we may regard $\psi : X \to G/H$ as a G-equivariant measurable map. Since $0 < \dim(\psi(x)) < \dim(\operatorname{Lie}(G))$ for ν-almost every $x \in X$, $w \in \operatorname{Gr}(\operatorname{Lie}(\mathbf{G}))_k$ is a nonzero proper subspace of $\operatorname{Lie}(\mathbf{G})_k$. Since $\mathbf{G}$ is a simple connected algebraic k-group, the adjoint algebraic k-representation $\operatorname{Ad} : \mathbf{G} \to \operatorname{GL}(\operatorname{Lie}(\mathbf{G}))$ is faithful irreducible and so $\mathbf{H} < \mathbf{G}$ is a proper k-subgroup. Denote by $\eta = \psi_*\nu \in \operatorname{Prob}(G/H)$ the pushforward μ-stationary Borel probability measure.

By Theorem 2.4 and [BS04, Corollary 5.2], the (G, μ)-space $(G/P, \nu_P)$ is the (G, μ)-Poisson boundary. Then there exists a G-equivariant measurable map $\beta : G/P \to \operatorname{Prob}(G/H)$ such that $\operatorname{Bar}(\beta_*\nu_P) = \eta$. We may assume that $\beta : G/P \to \operatorname{Prob}(G/H)$ is strictly G-equivariant. Then $\zeta = \beta(P) \in \operatorname{Prob}(G/H)$ is a P-invariant Borel probability measure. Since $P < \mathbf{P}$ is Zariski dense, there exists a normal k-subgroup $\mathbf{N} \lhd \mathbf{P}$ such that the image of P in $(\mathbf{P}/\mathbf{N})_k$ is compact and ζ is supported on $(\mathbf{G}/\mathbf{H})^{\mathbf{N}} \cap G/H$ (see [Sh97, Theorem 1.1] and [BDL14, Proposition 1.9]). Then $(\mathbf{G}/\mathbf{H})^{\mathbf{N}} \neq \emptyset$ and upon conjugating $\mathbf{H}$, we may assume that $\mathbf{N} < \mathbf{H}$. Since $N = \mathbf{N}_k < \mathbf{P}_k = P$ and $P < G$ are both cocompact, it follows that $N < G$ is cocompact and so is $H < G$.

Finally, we prove that $\mathbf{H}^0$ is not reductive. By contradiction, assume that $\mathbf{H}^0$ is reductive. Then the algebraic k-$\mathbf{G}$-variety $\mathbf{G}/\mathbf{H}$ is affine (see [Ha74, Theorem 3.3]). By [Bo91, Proposition 1.12], there exist an algebraic k-embedding $\iota : \mathbf{G}/\mathbf{H} \to \mathbf{V}$ into a finite dimensional k-vector space (that we may assume to have a spanning range) and an equivariant faithful algebraic k-representation $\rho : \mathbf{G} \to \operatorname{GL}(\mathbf{V})$. Since G leaves invariant the compact spanning subset $\iota(G/H) \subset \mathbf{V}_k$, it follows that G is compact, a contradiction. Therefore, $\mathbf{H}^0$ is not reductive. We may then consider its unipotent radical $\{e\} \neq R_u(\mathbf{H}^0) \lhd \mathbf{H}^0$, which is a k-subgroup. By [BT70, Proposition 3.1], there exists a proper parabolic k-subgroup $\mathbf{Q} < \mathbf{G}$ such that $\mathbf{H} < \mathbf{Q}$. Upon conjugating $\mathbf{Q}$, we may assume that $\mathbf{P} < \mathbf{Q} < \mathbf{G}$. Therefore, we obtain a G-equivariant measurable factor map $(X, \nu) \to (G/Q, \nu_Q)$. □

Proof of Theorem 3.2. By combining Theorems 3.3, 3.10 and (2.1) in case $k = \mathbb{R}$ (resp. [BBH21, Proposition 5.1] in case k is an arbitrary local field), there exist a proper parabolic k-subgroup $\mathbf{P} < \mathbf{Q} < \mathbf{G}$ and a normal unital $\ast$-embedding $\iota : \mathrm{L}^\infty(G/Q) \to \operatorname{mult}(\Theta)$ such that $\Theta \circ \iota : \mathrm{L}^\infty(G/Q) \hookrightarrow \mathrm{L}^\infty(G/P)$ is the canonical inclusion. Upon taking a smaller parabolic k-subgroup, we have $\operatorname{mult}(\Theta) \cong \mathrm{L}^\infty(G/Q)$ as G-von Neumann algebras. □

Let us point out that there is no higher rank assumption on the simple connected algebraic k-group $\mathbf{G}$ in Theorem 3.10. In particular, we obtain the following consequence.

Corollary 3.11. *Let k be a local field of characteristic zero. Let $\mathbf{G}$ be a simple connected algebraic k-group such that $\mathrm{rk}_k(\mathbf{G}) = 1$. Let $\mu \in \mathrm{Prob}(G)$ be an admissible Borel probability measure and (X, ν) an ergodic (G, μ)-space such that $G^+ \curvearrowright (X, \nu)$ is faithful. Assume that for ν-almost every $x \in X$, $\mathrm{Stab}_G(x)$ contains the group of k-points of a nontrivial k-split torus of $\mathbf{G}$.*

Then there exists a G-equivariant measurable factor map

$$(X, \nu) \to (G/P, \nu_P).$$

In particular, the action $G \curvearrowright (X, \nu)$ is amenable.

Proof. Since $\mathrm{rk}_k(\mathbf{G}) = 1$, the proper parabolic k-subgroup $\mathbf{P} < \mathbf{Q} < \mathbf{G}$ obtained by Theorem 3.10 satisfies $\mathbf{Q} = \mathbf{P}$ and so $Q = P$ is amenable. The existence of the G-equivariant measurable factor map $(X, \nu) \to (G/P, \nu_P)$ together with [Zi78, Proposition 3.1] imply that the action $G \curvearrowright (X, \nu)$ is amenable. □

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