Superrigidity

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Abstract. These are the lecture notes of a graduate course on *superrigidity* given at Université Paris-Saclay, Orsay, during 2022-2023. The aim of the course is to prove Margulis' superrigidity theorem for higher rank lattices (1975). We will present a recent proof due to Bader–Furman (2018) that relies on the concept of algebraic representations of ergodic actions. Topics include: locally compact groups and their lattices; ergodic group theory (metric ergodicity, amenability); introduction to algebraic groups (algebraic actions on algebraic varieties); algebraic representations of ergodic actions; Margulis' superrigidity theorem; Mostow–Margulis' rigidity theorem; Margulis' arithmeticity theorem.

Contents

CHAPTER 1

Locally compact groups

In this chapter, we give an introduction to the theory of locally compact groups and their lattices. We show that $\mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SL}_d(\mathbb{R})$ for every $d \geq 2$.

1. Generalities on locally compact groups

DEFINITION 1.1. Let G be a group endowed with a Hausdorff topology. We say that G is a *topological group* if the map $G \times G \to G : (g, h) \mapsto gh^{-1}$ is continuous. We then say that G is *locally compact* if there exists a compact neighborhood $U \subset G$ of the identity element $e \in G$.

Let G be a locally compact group. We say that G is

- *first countable* if there exists a countable neighborhood basis of $e \in G$.
- *second countable* if there exists a countable basis for the topology on G.
- *σ-compact* if there exists an increasing sequence of compact subsets $Q_n \subset G$ such that $G = \bigcup_{n \in \mathbb{N}} Q_n$.
- *compactly generated* if there exists a compact subset $Q \subset G$ such that $e \in Q$ and $G = \bigcup_{n \geq 1} Q^n$.
- *totally disconnected* if the connected component of $e \in G$ is equal to $\{e\}$.

The identity element $e \in G$ has a neighborhood basis consisting of compact subsets (see [**DE14**, Corollary A.8.2]). Any open subgroup $H < G$ is also closed since $G \setminus H = \bigcup_{gH \neq H} gH$. Any compactly generated group G is σ-compact. Any locally compact group G has a compactly generated open subgroup $H < G$. Indeed, choose a compact neighborhood $U \subset G$ of $e \in G$. Then $H = \bigcup_{n \geq 1} (U \cup U^{-1})^n$ is a compactly generated open subgroup of G. In particular, any *connected* locally compact group is compactly generated. A locally compact group G is second countable if and and only it is first countable and σ-compact (see [**St73**]). Moreover, any locally compact second countable group G is metrizable with a proper left invariant metric (see [**St73**]).

The class of locally compact groups is stable under taking closed subgroups, finite direct products and quotients with respect to closed normal subgroups. More precisely, we record the following facts.

PROPOSITION 1.2. *The following assertions hold:*

(i) *If G is a locally compact group and $H \leq G$ is a closed subgroup, then H endowed with the induced topology is locally compact.*
(ii) *If $d \geq 1$ and $G_1, \ldots, G_d$ are locally compact groups, then the product group $G = G_1 \times \cdots \times G_d$ endowed with the product topology is locally compact.*
(iii) *If G is a locally compact group and $N \lhd G$ is a closed normal subgroup, the quotient group G/N endowed with the quotient topology is locally compact.*
(iv) *If G is a locally compact group acting continuously on a locally compact group H by continuous automorphisms, then the semi-direct product group $G \ltimes H$ endowed with the product topology is locally compact.*

The proof of Proposition 1.2 is left to the reader as an exercise.

Examples 1.3. Here are some examples of locally compact groups. Let $d \geq 1$.

(i) Any group G endowed with the discrete topology is locally compact. In these notes, any countable group will always be endowed with its discrete topology.
(ii) Any compact group K is locally compact. In particular, the following compact groups

$$\mathbb{T}^d = \left\{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid \forall 1 \leq i \leq d, |z_i| = 1\right\}$$
$$\mathrm{SO}_d(\mathbb{R}) = \{A \in \mathrm{SL}_d(\mathbb{R}) \mid A^*A = AA^* = 1_d\}$$
$$\mathscr{U}(d) = \{A \in \mathrm{GL}_d(\mathbb{C}) \mid A^*A = AA^* = 1_d\}$$

are locally compact.
(iii) Any (finite dimensional) real Lie group G is locally compact.
  - The abelian group $(\mathbb{R}^d, +)$ endowed with the usual topology is locally compact.
  - The *general linear group* $\mathrm{GL}_d(\mathbb{R})$ can be regarded as the open (dense) subset of invertible matrices in $\mathrm{M}_d(\mathbb{R}) \cong \mathbb{R}^{d^2}$. Endowed with the topology coming from $\mathbb{R}^{d^2}$, the group $\mathrm{GL}_d(\mathbb{R})$ is locally compact.
  - The *special linear group* $\mathrm{SL}_d(\mathbb{R}) = \ker(\det)$ is a closed subgroup of $\mathrm{GL}_d(\mathbb{R})$ and so $\mathrm{SL}_d(\mathbb{R})$ is locally compact.
  - The semi-direct product group $\mathrm{SL}_d(\mathbb{R}) \ltimes \mathbb{R}^d$ is locally compact.
(iv) Any (finite dimensional) p-adic Lie group G is totally disconnected locally compact. In particular, for every prime $p \in \mathscr{P}$, the groups $\mathrm{GL}_d(\mathbb{Q}_p)$ and $\mathrm{SL}_d(\mathbb{Q}_p)$ are totally disconnected locally compact.
(v) Let $\mathsf{T} = (\mathsf{V}, \mathsf{E})$ be a locally finite tree and denote by $\mathrm{Aut}(\mathsf{T})$ the automorphism group of T. Endowed with the topology of pointwise convergence, the group $\mathrm{Aut}(\mathsf{T})$ is totally disconnected locally compact.

Let X be a locally compact space, meaning that every $x \in X$ has a compact neighborhood. We denote by $\mathscr{B}(X)$ the σ-algebra of Borel subsets of X. We say that a Borel measure ν on X, that is, a measure defined on $\mathscr{B}(X)$ is *regular* if the following conditions are satisfied:

(i) For every Borel subset $B \subset X$, we have

$$\nu(B) = \inf \{\nu(V) \mid V \text{ is open and } B \subset V\}.$$

(ii) For every open subset $U \subset X$, we have

$$\nu(U) = \sup \{\nu(K) \mid K \text{ is compact and } K \subset U\}.$$

(iii) For every compact subset $K \subset X$, we have $\nu(K) < +\infty$.

When ν is nonzero, define the *support* of ν by

$$\operatorname{supp}(\nu) = \bigcap \{F \mid F \subset X \text{ is closed and } \nu(X \setminus F) = 0\}.$$

Observe that $\operatorname{supp}(\nu)$ is closed and $\nu(X \setminus \operatorname{supp}(\nu)) = 0$.

If any open subset of X is σ-compact, then any Borel measure on X that satisfies condition (iii) is regular (see [**Ru87**, Theorem 2.18]). In particular, using [**DE14**, Lemma A.8.1(i)], if X is a locally compact second countable space, then any open subset of X is σ-compact and thus any Borel measure on X that satisfies condition (iii) is regular.

Denote by $\mathrm{C}_c(X)$ the space of compactly supported continuous functions on X. We say that a linear functional $\Phi : \mathrm{C}_c(X) \to \mathbb{C}$ is *positive* if $\Phi(f) \geq 0$ for every $f \in \mathrm{C}_c(X)_+$. By Riesz's representation theorem (see [**Ru87**, Theorem 2.14]), for every positive linear functional $\Phi : \mathrm{C}_c(X) \to \mathbb{C}$, there exists a unique regular Borel measure ν on X such that

$$\forall f \in \mathrm{C}_c(X), \quad \Phi(f) = \int_X f(x) \, \mathrm{d}\nu(x).$$

In that case, we will simply write $\Phi = \nu$. Note that for every regular Borel measure ν on X and every $p \in [1, +\infty)$, the space $\mathrm{C}_c(X)$ is $\| \cdot \|_p$-dense in the Banach space $\mathrm{L}^p(X, \mathscr{X}, \nu)$ of all ν-equivalence classes of p-integrable functions on X.

THEOREM 1.4 (Haar). *Let G be a locally compact group. Then there exists a nonzero regular Borel measure m_G on G that is unique up to multiplicative constant and that satisfies one of the following equivalent conditions:*

(i) *For every Borel subset $B \subset G$ and every $g \in G$, $m_G(gB) = m_G(B)$.*

(ii) *For every $f \in \mathrm{C}_c(G)$ and every $g \in G$,*

$$\int_G f(g^{-1}h) \, \mathrm{d}m_G(h) = \int_G f(h) \, \mathrm{d}m_G(h)$$

We say that m_G is a left invariant Haar measure *on G.*

For a proof of Theorem 1.4, we refer the reader to [**HR79**, Chapter 15]. The locally compact group G is σ-compact if and only if the left invariant Haar measure m_G is σ-finite.

Theorem 1.4 also implies that there exists a nonzero regular Borel measure μ_G on G that is unique up to multiplicative constant and that satisfies one of the following equivalent conditions:

(i) For every Borel subset $B \subset G$ and every $g \in G$, $\mu_G(Bg) = \mu_G(B)$.
(ii) For every $f \in \mathrm{C}_c(G)$ and every $g \in G$,

$$\int_G f(hg)\,\mathrm{d}\mu_G(h) = \int_G f(h)\,\mathrm{d}\mu_G(h)$$

We say that μ_G is a *right invariant Haar measure* on G. Indeed, any left invariant Haar measure m_G on G gives rise to a right invariant Haar measure μ_G on G by the formula

$$\forall B \in \mathscr{B}(G), \quad \mu_G(B) = m_G(B^{-1}).$$

The next proposition shows that any left invariant Haar measure has full support.

Proposition 1.5. *Let G be a locally compact group and m_G a left invariant Haar measure on G. Then* $\mathrm{supp}(m_G) = G$. *Moreover, for every $f \in \mathrm{C}_c(G)_+$ such that $f \neq 0$, we have $\int_G f(h)\,\mathrm{d}m_G(h) > 0$.*

Proof. Since $m_G \neq 0$, Conditions (ii) and (iii) in the definition of regularity imply that there exists a compact subset $K \subset G$ such that $0 < m_G(K) < +\infty$. Let $U \subset G$ be a nonempty open subset. There exist $g_1, \dots, g_n \in G$ such that $K \subset \bigcup_{i=1}^n g_i U$. This implies that

$$0 < m_G(K) \leq m_G(\bigcup_{i=1}^n g_i U) \leq \sum_{i=1}^n m_G(g_i U) = n \cdot m_G(U)$$

and so $m_G(U) > 0$. Thus, $\mathrm{supp}(m_G) = G$.

Moreover, let $f \in \mathrm{C}_c(G)_+$ such that $f \neq 0$. Then there exist $\varepsilon > 0$ and an open subset $U \subset G$ such that $f(h) \geq \varepsilon$ for every $h \in U$. This implies that

$$\int_G f(h)\,\mathrm{d}m_G(h) \geq \int_U \varepsilon\,\mathrm{d}m_G(h) = \varepsilon \cdot m_G(U) > 0.$$

This finishes the proof. □

The next proposition gives a characterization of compact groups in terms of the Haar measure.

Proposition 1.6. *Let G be a locally compact group and m_G a left invariant Haar measure on G.*

Then G is compact if and only if $m_G(G) < +\infty$.

Proof. Firstly, assume that G is compact. Then by regularity we have $m_G(G) < +\infty$.

Secondly, assume that G is not compact. Take a compact neighborhood $K \subset G$ of $e \in G$ and set $g_0 = e$. We have $m_G(K) > 0$ by Proposition 1.5. Since KK^{-1} is compact, there exists $g_1 \in G$ such that $g_1 \in G \setminus KK^{-1}$. This implies that $g_1 K \cap K = \emptyset$. By induction, define $g_n \in G$ so that

$g_n \in G \setminus (K \cup g_1 K \cup \cdots \cup g_{n-1} K) K^{-1}$. It follows that $(g_n K)_n$ are pairwise disjoint. This implies that

$$m_G(G) \geq m_G(\bigcup_{n \in \mathbb{N}} g_n K) = \sum_{n \in \mathbb{N}} m_G(g_n K) = +\infty \cdot m_G(K) = +\infty.$$

This finishes the proof. □

Let G be a locally compact group and m_G a left invariant Haar measure on G. The measure m_G need not be right invariant. For every $g \in G$, define the nonzero regular Borel measure m_G^g on G by the formula $m_G^g(B) = m_G(Bg)$ for every $B \in \mathscr{B}(G)$. Since m_G^g is a left invariant Haar measure, there exists an element $\Delta_G(g) \in \mathbb{R}_+^*$ such that $m_G^g = \Delta_G(g)\, m_G$. Then $\Delta_G : G \to \mathbb{R}_+^* : g \mapsto \Delta_G(g)$ is a group homomorphism and is called the *modular function* on G. The modular function Δ_G does not depend on the choice of the left invariant Haar measure m_G on G. Moreover, we have

$$\forall f \in \mathrm{C}_c(G), \forall g \in G, \quad \int_G f(hg^{-1})\, \mathrm{d}m_G(h) = \Delta_G(g) \int_G f(h)\, \mathrm{d}m_G(h). \tag{1.1}$$

The left invariant Haar measure m_G is right invariant if and only if $\Delta_G \equiv 1$. In that case, we say that G is *unimodular*. We then simply refer to m_G as a Haar measure on G.

Proposition 1.7. *Let G be a locally compact group and m_G a left invariant Haar measure on G. Then the modular function $\Delta_G : G \to \mathbb{R}_+^*$ is continuous. Moreover, we have*

$$\forall f \in \mathrm{C}_c(G), \quad \int_G f(h^{-1})\, \mathrm{d}m_G(h) = \int_G \Delta_G(h^{-1}) f(h)\, \mathrm{d}m_G(h).$$

Proof. Choose $\varphi \in \mathrm{C}_c(G)$ such that $\kappa = \int_G \varphi(h)\, \mathrm{d}m_G(h) \neq 0$. Set $Q = \operatorname{supp}(\varphi)$. Then we have

$$\forall g \in G, \quad \Delta_G(g) = \frac{\int_G \varphi(hg^{-1})\, \mathrm{d}m_G(h)}{\int_G \varphi(h)\, \mathrm{d}m_G(h)}.$$

Choose a compact neighborhood $K \subset G$ of $e \in G$. Let $\varepsilon > 0$. Since φ is uniformly continuous by Lemma 1.8, there exists a neighborhood U of $e \in G$ such that $U \subset K$, $U^{-1} = U$ and

$$\forall u \in U, \quad \sup\left\{|\varphi(hu^{-1}) - \varphi(h)| \mid h \in G\right\} \leq \frac{\varepsilon\kappa}{m_G(QK)}.$$

Then for every $u \in U$, we have

$$\begin{aligned} |\Delta_G(u) - 1| &\leq \frac{1}{\kappa} \int_G |\varphi(hu^{-1}) - \varphi(h)|\, \mathrm{d}m_G(h) \\ &\leq \frac{1}{\kappa} m_G(QK) \frac{\varepsilon\kappa}{m_G(QK)} = \varepsilon. \end{aligned}$$

This implies that $\Delta_G : G \to \mathbb{R}_+^*$ is continuous at the identity element $e \in G$ and so Δ_G is continuous.

Next, observe that both of the positive linear functionals

$$\mathrm{C}_c(G) \to \mathbb{C} : f \mapsto \int_G f(h^{-1})\,\mathrm{d}m_G(h)$$
$$\mathrm{C}_c(G) \to \mathbb{C} : f \mapsto \int_G \Delta(h^{-1}) f(h)\,\mathrm{d}m_G(h)$$

define a nonzero right invariant regular Borel measure on G. Thus, there exists $c > 0$ such that

$$\forall f \in \mathrm{C}_c(G), \quad \int_G f(h^{-1})\,\mathrm{d}m_G(h) = c \int_G \Delta_G(h^{-1}) f(h)\,\mathrm{d}m_G(h)$$

Define $\widehat{\varphi} \in \mathrm{C}_c(G)$ by the formula $\widehat{\varphi}(h) = \varphi(h^{-1})$ for every $h \in G$. Then we have

$$\begin{aligned} 0 \neq \int_G \varphi(h)\,\mathrm{d}m_G(h) &= \int_G \widehat{\varphi}(h^{-1})\,\mathrm{d}m_G(h) \\ &= c \int_G \Delta_G(h^{-1})\widehat{\varphi}(h)\,\mathrm{d}m_G(h) \\ &= c \int_G \Delta_G(h^{-1})\varphi(h^{-1})\,\mathrm{d}m_G(h) \\ &= c^2 \int_G \Delta_G(h^{-1})\Delta_G(h)\varphi(h)\,\mathrm{d}m_G(h) \\ &= c^2 \int_G \varphi(h)\,\mathrm{d}m_G(h). \end{aligned}$$

This implies that $c = 1$. □

In the proof of Proposition 1.7, we used the following technical result. Denote by $(\mathrm{C}_b(G), \|\cdot\|_\infty)$ the Banach space of all bounded continuous functions on G endowed with the supremum norm. Denote by $\lambda : G \curvearrowright \mathrm{C}_b(G)$ (resp. $\rho : G \curvearrowright \mathrm{C}_b(G)$) the left (resp. right) translation action defined by $(\lambda(g)f)(h) = f(g^{-1}h)$ (resp. $(\rho(g)f)(h) = f(hg)$) for all $g, h \in G$ and all $f \in \mathrm{C}_b(G)$.

Lemma 1.8. *Let G be a locally compact group and $f \in \mathrm{C}_c(G)$ a compactly supported continuous function. Then for every $\varepsilon > 0$, there exists a symmetric neighborhood $U \subset G$ of $e \in G$ such that*

$$\sup\{\|\lambda(u)f - f\|_\infty, \|\rho(u)f - f\|_\infty \mid u \in U\} < \varepsilon.$$

Then we say that $f \in \mathrm{C}_c(G)$ is uniformly continuous.

Proof. Let $f \in \mathrm{C}_c(G)$ and set $Q = \mathrm{supp}(f)$. Let $\varepsilon > 0$ and fix a symmetric compact neighborhood $V \subset G$ of $e \in G$. For every $g \in G$, there exists an open neighborhood $W_g \subset G$ of $g \in G$ such that for all $w_1, w_2 \in W_g$, we have $|f(w_1) - f(w_2)| < \varepsilon$. For every $g \in G$, choose an open symmetric neighborhood $U_g \subset G$ of $e \in G$ such that $gU_gU_g \cup U_gU_gg \subset W_g$. Then for every $g \in G$, $gU_g \cap U_gg$ is an open neighborhood of $g \in G$. Since VQV is compact, there exist $n \geq 1$ and $g_1, \ldots, g_n \in G$ such that

$VQV \subset \bigcup_{i=1} g_i U_{g_i} \cap U_{g_i} g_i$. Define $U = V \cap \bigcap_{i=1}^n U_{g_i}$ which is a symmetric neighborhood of the identity $e \in G$. Then for every $u \in U$ and every $g \in G$, we consider the following situations:

- If $g \in VQV$, then there exists $1 \leq i \leq n$ such that $g \in g_i U_{g_i} \cap U_{g_i} g_i$. Since $u \in U \subset U_{g_i}$, we have $gu \in g_i U_{g_i} U_{g_i} \subset W_{g_i}$ and $ug \in U_{g_i} U_{g_i} g_i \subset W_{g_i}$. It follows that $|f(gu) - f(g)| < \varepsilon$ and $|f(gu) - f(g)| < \varepsilon$.
- If $g \notin VQV$, then $gu \notin Q$ and $ug \notin Q$. It follows that $f(g) = f(ug) = f(gu) = 0$.

We have showed that for every $u \in U$ and every $g \in G$, we have $|f(gu) - f(g)| < \varepsilon$ and $|f(gu) - f(g)| < \varepsilon$. □

Let (G, m_G, Δ_G) and (H, m_H, Δ_H) be locally compact groups with their respective left invariant Haar measure and modular function. Let $\sigma : G \curvearrowright H$ be a continuous action by continuous group automorphisms and write $G \ltimes H$ for the locally compact semi-direct product group. Recall that the group law on $G \ltimes H$ is given by

$$\forall g_1, g_2 \in G, \forall h_1, h_2 \in H, \quad (g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, \sigma_{g_2}^{-1}(h_1) h_2).$$

The next proposition provides an explicit calculation of the Haar measure and the modular function on $G \ltimes H$.

Proposition 1.9. *The regular Borel measure $m_{G \ltimes H}$ defined on $G \ltimes H$ by the formulae*

$$\begin{aligned} \forall f \in \mathrm{C}_c(G \ltimes H), \int_{G \ltimes H} & f(g,h)\, \mathrm{d}m_{G \ltimes H}(h) \\ &= \int_H \left(\int_G f(g,h)\, \mathrm{d}m_G(g) \right) \mathrm{d}m_H(h) \\ &= \int_G \left(\int_H f(g,h)\, \mathrm{d}m_H(h) \right) \mathrm{d}m_G(g) \end{aligned} \tag{1.2}$$

is a left invariant Haar measure on $G \ltimes H$. Moreover, the modular function $\Delta_{G \ltimes H} : G \ltimes H \to \mathbb{R}_+^*$ satisfies

$$\forall (g,h) \in G \ltimes H, \quad \Delta_{G \ltimes H}(g,h) = \rho(g)\, \Delta_G(g)\, \Delta_H(h)$$

where $\rho : G \to \mathbb{R}_+^*$ is the continuous function defined by the formula

$$\forall f \in \mathrm{C}_c(H), \forall g \in G, \quad \int_H f(\sigma_g(h))\, \mathrm{d}m_H(h) = \rho(g) \int_H f(h)\, \mathrm{d}m_H(h).$$

Proof. Fubini's theorem implies that for every $f \in \mathrm{C}_c(G \ltimes H)$, we have

$$\int_H \left(\int_G f(g,h)\, \mathrm{d}m_G(g) \right) \mathrm{d}m_H(h) = \int_G \left(\int_H f(g,h)\, \mathrm{d}m_H(h) \right) \mathrm{d}m_G(g).$$

Denote by $m_{G\ltimes H}$ the unique regular Borel measure on $G \ltimes H$ defined by (1.2). For every $f \in \mathrm{C}_c(G \ltimes H)$ and every $(g_1, h_1) \in G \ltimes H$, we have

$$\begin{aligned}
&\int_{G\ltimes H} f((g_1,h_1)\cdot(g_2,h_2))\,\mathrm{d}m_{G\ltimes H}(g_2,h_2)\\
&\qquad = \int_{G\ltimes H} f(g_1g_2, \sigma_{g_2}^{-1}(h_1)h_2)\,\mathrm{d}m_{G\ltimes H}(g_2,h_2)\\
&\qquad = \int_G \left(\int_H f(g_1g_2,h_2)\,\mathrm{d}m_H(h_2)\right)\mathrm{d}m_G(g_2)\\
&\qquad = \int_H \left(\int_G f(g_2,h_2)\,\mathrm{d}m_G(g_2)\right)\mathrm{d}m_H(h_2)\\
&\qquad = \int_{G\ltimes H} f(g_2,h_2)\,\mathrm{d}m_{G\ltimes H}(g_2,h_2).
\end{aligned}$$

This shows that $m_{G\ltimes H}$ is a left invariant Haar measure on $G \ltimes H$.

Consider the function $\rho : G \to \mathbb{R}_+^*$ as defined above. For every $f \in \mathrm{C}_c(G \ltimes H)$ and every $(g_2, h_2) \in G \ltimes H$, we have

$$\begin{aligned}
&\int_{G\ltimes H} f((g_1,h_1)\cdot(g_2,h_2)^{-1})\,\mathrm{d}m_{G\ltimes H}(g_1,h_1)\\
&\quad = \int_{G\ltimes H} f(g_1g_2^{-1}, \sigma_{g_2}(h_1h_2^{-1}))\,\mathrm{d}m_{G\ltimes H}(g_1,h_1)\\
&\quad = \Delta_H(h_2)\int_G \left(\int_H f(g_1g_2^{-1},\sigma_{g_2}(h_1))\,\mathrm{d}m_H(h_1)\right)\mathrm{d}m_G(g_1)\\
&\quad = \rho(g_2)\,\Delta_H(h_2)\int_G \left(\int_H f(g_1g_2^{-1},h_1)\,\mathrm{d}m_H(h_1)\right)\mathrm{d}m_G(g_1)\\
&\quad = \rho(g_2)\,\Delta_G(g_2)\,\Delta_H(h_2)\int_H \left(\int_G f(g_1,h_1)\,\mathrm{d}m_G(g_1)\right)\mathrm{d}m_H(h_1)\\
&\quad = \rho(g_2)\,\Delta_G(g_2)\,\Delta_H(h_2)\int_{G\ltimes H} f(g_1,h_1)\,\mathrm{d}m_{G\ltimes H}(g_1,h_1)
\end{aligned}$$

and hence $\Delta_{G\ltimes H}(g_2,h_2) = \rho(g_2)\,\Delta_G(g_2)\,\Delta_H(h_2)$. □

EXAMPLES 1.10. Here are some examples of unimodular locally compact groups. Let $d \geq 1$.

(i) Any group G endowed with the discrete topology is unimodular. Indeed, in that case the counting measure m_G is a nonzero regular Borel measure on G that is clearly both left and right invariant.
(ii) Any compact group G is unimodular. Indeed, fix a left invariant Haar measure m_G on G. Then $\Delta_G(G) < \mathbb{R}_+^*$ is a compact subgroup and so $\Delta_G(G) = \{1\}$. This shows that $\Delta_G \equiv 1$ and so G is unimodular.
(iii) Any abelian locally compact group G is unimodular. The Lebesgue measure $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ on $\mathbb{R}^d$ is a Haar measure.

(iv) Recall that the general linear group $\mathrm{GL}_d(\mathbb{R})$ can be regarded as the open (dense) subset of invertible matrices in $\mathrm{M}_d(\mathbb{R}) \cong \mathbb{R}^d \times \cdots \times \mathbb{R}^d$. For every $g \in \mathrm{GL}_d(\mathbb{R})$, the Jacobian of the diffeomorphism

$$L_g : \mathrm{M}_d(\mathbb{R}) \to \mathrm{M}_d(\mathbb{R}) : (x_1, \ldots, x_d) \mapsto (gx_1, \ldots, gx_d)$$

is equal to $|\det(g)|^d$. It follows that a left invariant Haar measure m_G on $G = \mathrm{GL}_d(\mathbb{R})$ is given by

$$\mathrm{d}m_G(g) = \frac{1}{|\det(g)|^d} \prod_{1 \leq i,j \leq d} \mathrm{d}g_{ij}, \quad g = (g_{ij})_{ij}.$$

For every $g \in \mathrm{GL}_d(\mathbb{R})$, since the Jacobian of the diffeomorphism

$$R_g : \mathrm{M}_d(\mathbb{R}) \to \mathrm{M}_d(\mathbb{R}) : x \mapsto xg$$

is also equal to $|\det(g)|^d$, it follows that m_G is right invariant and so $G = \mathrm{GL}_d(\mathbb{R})$ is unimodular.

(v) Recall that the special linear group $\mathrm{SL}_d(\mathbb{R}) < \mathrm{GL}_d(\mathbb{R})$ is defined by $\mathrm{SL}_d(\mathbb{R}) = \ker(\det)$. It is known that the only normal subgroups of $\mathrm{SL}_d(\mathbb{R})$ are $\{1\}$, $\{\pm 1\}$ and $\mathrm{SL}_d(\mathbb{R})$. This implies that $\ker(\Delta_{\mathrm{SL}_d(\mathbb{R})}) = \mathrm{SL}_d(\mathbb{R})$ and so $\mathrm{SL}_d(\mathbb{R})$ is unimodular.

(vi) For every $d \geq 2$, the strict upper triangular subgroup $G = \mathrm{T}_d(\mathbb{R})$ defined as the group of all matrices $g = (g_{ij})_{ij}$ such that $g_{ij} = 0$ for all $1 \leq j < i \leq d$ and $g_{ii} = 1$ for all $1 \leq i \leq d$ is homeomorphic with $\mathbb{R}^{\frac{d(d-1)}{2}}$. Under this identification, the Lebesgue measure on $\mathbb{R}^{\frac{d(d-1)}{2}}$ gives rise to a left and right invariant Haar measure m_G on G defined as

$$\mathrm{d}m_G(n) = \prod_{1 \leq i < j \leq d} \mathrm{d}n_{ij}, \quad n = (n_{ij})_{ij}.$$

Indeed, for all $i < j$ and all $g, n \in \mathrm{T}_d(\mathbb{R})$, we have $(gn)_{ij} = g_{ij} + n_{ij} + \sum_{i<k<j} g_{ik} n_{kj}$. Endow the set $\{(i,j) \mid 1 \leq i < j \leq d\}$ with the lexicographical order. Then for every $g \in \mathrm{T}_d(\mathbb{R})$, the Jacobian matrix of the diffeomorphism $\mathrm{T}_d(\mathbb{R}) \to \mathrm{T}_d(\mathbb{R}) : n \mapsto gn$ is lower triangular with diagonal entries all equal to 1. This implies that the Jacobian of the diffeomorphism $\mathrm{T}_d(\mathbb{R}) \to \mathrm{T}_d(\mathbb{R}) : n \mapsto gn$ is equal to 1. The same argument shows that for every $g \in \mathrm{T}_d(\mathbb{R})$, the Jacobian of the diffeomorphism $\mathrm{T}_d(\mathbb{R}) \to \mathrm{T}_d(\mathbb{R}) : n \mapsto ng$ is equal to 1. Thus, $G = \mathrm{T}_d(\mathbb{R})$ is unimodular.

2. Lattices in locally compact groups

Let G be a locally compact group and $\Gamma < G$ a discrete subgroup. We say that a Borel subset $\mathscr{F} \subset G$ is a *Borel fundamental domain* (for the right translation action $\Gamma \curvearrowright G$) if

$$\forall \gamma_1, \gamma_2 \in \Gamma,\ \gamma_1 \neq \gamma_2 \ \Rightarrow\ \mathscr{F}\gamma_1 \cap \mathscr{F}\gamma_2 = \emptyset \quad \text{and} \quad \bigcup_{\gamma \in \Gamma} \mathscr{F}\gamma = G.$$

Denote by $G/\Gamma = \{g\Gamma \mid g \in G\}$ the quotient space and by $p : G \to G/\Gamma : g \mapsto g\Gamma$ the quotient map. Endow G/Γ with the quotient topology.

Proposition 1.11. *Keep the same notation as above. The following assertions hold:*

(i) *The quotient map $p : G \to G/\Gamma$ is continuous and open and G/Γ is Hausdorff and locally compact. Moreover, the action map $G \times G/\Gamma \to G/\Gamma : (g, x) \mapsto gx$ is continuous.*

(ii) *If G/Γ is compact, then there exists a Borel fundamental domain $\mathscr{F} \subset G$ that is relatively compact in G.*

(iii) *If G is second countable, then G/Γ is second countable. Moreover, there exists a Borel fundamental domain $\mathscr{F} \subset G$ such that for every compact subset $Y \subset G/\Gamma$, the subset $p^{-1}(Y) \cap \mathscr{F} \subset G$ is relatively compact in G.*

Proof. (i) Endow the quotient space $G/\Gamma = \{g\Gamma \mid g \in G\}$ with the quotient topology. By definition, a subset $V \subset G/\Gamma$ is open if and only if $p^{-1}(V) \subset G$ is open. Then the quotient topology is the finest topology on G/Γ that makes the quotient map $p : G \to G/\Gamma$ continuous. Let now $U \subset G$ be an open set. Then $p^{-1}(p(U)) = p^{-1}(\{h\Gamma \mid h \in U\}) = \bigcup_{\gamma \in \Gamma} U\gamma$ is open and so is $p(U) \subset G/\Gamma$ is open. This shows that $p : G \to G/\Gamma$ is open.

Let $x_1, x_2 \in G/\Gamma$ with $x_1 \neq x_2$. Write $x_1 = g_1\Gamma$ and $x_2 = g_2\Gamma$. Note that $g_2 \notin g_1\Gamma$. Choose a compact neighborhood $U_1 \subset G$ (resp. $U_2 \subset G_2$) of $g_1 \in G$ (resp. $g_2 \in G$). Since $U_2^{-1}U_1 \subset G$ is compact and since $\Gamma < G$ is discrete, the set $\Lambda = \{\gamma \in \Gamma \mid U_1 \cap U_2\gamma \neq \emptyset\}$ is finite. For every $\gamma \in \Lambda$, since $g_1 \neq g_2\gamma$, there exist neighborhoods U_γ of $g_1 \in G$ and V_γ of $g_2\gamma \in G$ such that $U_\gamma \cap V_\gamma = \emptyset$. Set

$$W_1 = U_1 \cap \bigcap_{\gamma \in \Lambda} U_\gamma \quad \text{and} \quad W_2 = U_2 \cap \bigcap_{\gamma \in \Lambda} V_\gamma\gamma^{-1}.$$

Then for every $\gamma \in \Gamma$, we have $W_1 \cap W_2\gamma = \emptyset$. Indeed, if $\gamma \in \Gamma \setminus \Lambda$, then $U_1 \cap U_2\gamma = \emptyset$. If $\gamma \in \Lambda$, then $U_\gamma \cap (V_\gamma\gamma^{-1})\gamma = \emptyset$. Thus, we have $p(W_1) \cap p(W_2) = \emptyset$. This shows that G/Γ is Hausdorff.

Let $x = g\Gamma \in G/\Gamma$. Choose a compact neighborhood $K \subset G$ of $e \in G$. Then gK is a compact neighborhood of $g \in G$ and so $p(gK)$ is a compact neighborhood of $x \in G/\Gamma$. This shows that G/Γ is locally compact.

Define the action map $a : G \times G/\Gamma \to G/\Gamma : (g, x) \mapsto gx$. Recall that the multiplication map $m : G \times G \to G$ is continuous. Since the map $\mathrm{id}_G \times p : G \times G \to G \times G/\Gamma : (g, h) \mapsto (g, h\Gamma)$ is continuous and open, the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\ m\ } & G \\ \big\downarrow{\scriptstyle \mathrm{id} \times p} & & \big\downarrow{\scriptstyle p} \\ G \times G/\Gamma & \xrightarrow{\ a\ } & G/\Gamma \end{array}$$

shows that the action map $a : G \times G/\Gamma \to G/\Gamma$ is continuous.

(ii) Since $\Gamma < G$ is discrete, there exists an open neighborhood $V \subset G$ of $e \in G$ such that $V \cap \Gamma = \{e\}$. Since the map $G \times G \to G : (g, h) \mapsto g^{-1}h$ is continuous, there exists an open neighborhood $U \subset G$ of $e \in G$ such that $U^{-1}U \subset V$. Replacing U with $U \cap K$ where K is a relatively compact open neighborhood of $e \in G$, we may assume that $U \subset G$ is relatively compact. Since G/Γ is compact and since $(p(gU)_{g\in G})$ is an open covering of G/Γ, there exist $g_1, \dots, g_n \in G$ such that $G/\Gamma = \bigcup_{i=1}^n p(g_iU)$. Define the Borel subset

$$\mathscr{F} = \bigcup_{i=1}^{n} \left(g_iU \setminus \bigcup_{j<i} g_jU\Gamma \right).$$

By construction, $\mathscr{F} \subset G$ is relatively compact. Then we have $\bigcup_{\gamma\in\Gamma} \mathscr{F}\gamma = \bigcup_{i=1}^n g_iU\Gamma = p^{-1}(\bigcup_{i=1}^n p(g_iU)) = p^{-1}(G/\Gamma) = G$. Let $\gamma_1, \gamma_2 \in \Gamma$ be elements such that $\mathscr{F}\gamma_1 \cap \mathscr{F}\gamma_2 \neq \emptyset$. Upon exchanging γ_1 and γ_2, we may assume that there exist $i \geq j$ and $u_1, u_2 \in U$ such that $g_iu_1\gamma_1 = g_ju_2\gamma_2$. By construction and since $g_iu_1 = g_ju_2\gamma_2\gamma_1^{-1} \in g_iU \cap g_jU\Gamma$, we necessarily have $i = j$. Then $u_1\gamma_1 = u_2\gamma_2$ and so $u_2^{-1}u_1 = \gamma_2\gamma_1^{-1} \in U^{-1}U \cap \Gamma \subset V \cap \Gamma = \{e\}$. This shows that $\gamma_1 = \gamma_2$ and thus $\mathscr{F} \subset G$ is a Borel fundamental domain.

(iii) Choose a countable basis $(U_n)_n$ for the topology on G. Let $V \subset G/\Gamma$ be an open set. Then $p^{-1}(V) = \bigcup_{\gamma\in\Gamma} V\gamma \subset G$ is open and so there exists a subfamily $(U_{n_k})_k$ such that $p^{-1}(V) = \bigcup_k U_{n_k}$. Then we have $V = p(p^{-1}(V)) = \bigcup_k p(U_{n_k})$. This shows that $(p(U_n))_n$ is a countable basis for the quotient topology on G/Γ and so G/Γ is second countable. For every $n \in \mathbb{N}$, choose $g_n \in U_n$.

As before, there exist open neighborhoods $U, V \subset G$ of $e \in G$ such that $U \subset G$ is relatively compact, $U^{-1}U \subset V$ and $V \cap \Gamma = \{e\}$. We claim that $G = \bigcup_{n\in\mathbb{N}} g_nU$. Indeed, for every $g \in G$, $gU^{-1} \subset G$ is an open set and hence there exists $n \in \mathbb{N}$ such that $U_n \subset gU^{-1}$. This implies that there exists $u \in U$ such that $g_n = gu^{-1}$ or equivalently $g = g_nu$ and thus $g \in g_nU$. Define the Borel subset

$$\mathscr{F} = \bigcup_{n\in\mathbb{N}} \left(g_nU \setminus \bigcup_{k<n} g_kU\Gamma \right).$$

Then we have $\bigcup_{\gamma\in\Gamma} \mathscr{F}\gamma = \bigcup_{n\in\mathbb{N}} g_nU\Gamma = G$. Let $\gamma_1, \gamma_2 \in \Gamma$ be elements such that $\mathscr{F}\gamma_1 \cap \mathscr{F}\gamma_2 \neq \emptyset$. Upon exchanging γ_1 and γ_2, we may assume that there exist $m \geq n$ and $u_1, u_2 \in U$ such that $g_mu_1\gamma_1 = g_nu_2\gamma_2$. By construction and since $g_mu_1 = g_nu_2\gamma_2\gamma_1^{-1} \in g_mU \cap g_nU\Gamma$, we necessarily have $m = n$. Then $u_1\gamma_1 = u_2\gamma_2$ and so $u_2^{-1}u_1 = \gamma_2\gamma_1^{-1} \in U^{-1}U \cap \Gamma \subset V \cap \Gamma = \{e\}$. This shows that $\gamma_1 = \gamma_2$ and thus $\mathscr{F} \subset G$ is a Borel fundamental domain. Let $Y \subset G/\Gamma$ be a compact subset. Since $(p(g_nU))_n$ is an open covering of Y, there exist $n_1 \leq \cdots \leq n_k$ such that $Y \subset \bigcup_{i=1}^k p(g_{n_i}U)$. Then we have $p^{-1}(Y) \cap \mathscr{F} \subset \bigcup_{j=0}^{n_k}(g_jU \setminus \bigcup_{i<j} g_iU\Gamma)$ and so $p^{-1}(Y) \cap \mathscr{F} \subset G$ is relatively compact. $\square$

Observe that when G is a locally compact σ-compact group, any discrete subgroup $\Gamma < G$ is necessarily countable. Indeed, since G is σ-compact, the left invariant Haar measure m_G is σ-finite. We may then choose a Borel probability measure $\mu \in \mathrm{Prob}(G)$ such that $\mu \sim m_G$. We may also choose open neighborhoods $U, V \subset G$ of $e \in G$ such that $UU^{-1} \subset V$ and $V \cap \Gamma = \{e\}$. Then $(\gamma U)_{\gamma \in \Gamma}$ is a family of pairwise disjoint open subsets. Moreover, since $m_G(\gamma U) = m_G(U) > 0$ for every $\gamma \in \Gamma$, it follows that $\mu(\gamma U) > 0$ for every $\gamma \in \Gamma$. This implies that Γ is necessarily countable.

Corollary 1.12. *Let G be a locally compact second countable group and $\Gamma < G$ a discrete subgroup. Then there exists a Borel map $\sigma : G/\Gamma \to G$ such that*

- *$\sigma(G/\Gamma) = \mathscr{F}$ is a Borel fundamental domain,*
- *$\sigma(\Gamma) = e$,*
- *$x = \sigma(x)\Gamma$ for every $x \in G/\Gamma$,*
- *$\sigma(Y) \subset G$ is relatively compact for every compact subset $Y \subset G/\Gamma$.*

We then simply say that $\sigma : G/\Gamma \to G$ is a Borel section*.*

Proof. Choose a Borel fundamental domain $\mathscr{F} \subset G$ as in Proposition 1.11(iii) such that $e \in \mathscr{F}$. Then $p|_{\mathscr{F}} : \mathscr{F} \to G/\Gamma$ is Borel and bijective. This implies that the map $\sigma = (p|_{\mathscr{F}})^{-1} : G/\Gamma \to G$ is Borel (see [**Zi84**, Theorem A.4]) and satisfies all the required properties. □

Definition 1.13. Let G be a locally compact group and $\Gamma < G$ a discrete subgroup. We say that $\Gamma < G$ is *uniform* or *cocompact* if G/Γ is compact.

We say that $\Gamma < G$ is a *lattice* if there exists a G-invariant regular Borel probability measure $\nu \in \mathrm{Prob}(G/\Gamma)$.

Define the linear mapping $\mathscr{T} : \mathrm{C}_c(G) \to \mathrm{C}_c(G/\Gamma) : f \mapsto \overline{f}$ by the formula

$$\forall g \in G, \quad \overline{f}(g\Gamma) = \sum_{\gamma \in \Gamma} f(g\gamma).$$

We claim that $\mathscr{T} : \mathrm{C}_c(G) \to \mathrm{C}_c(G/\Gamma)$ is surjective. Indeed, let $\varphi \in \mathrm{C}_c(G/\Gamma)$ be a function and denote by $Q = \mathrm{supp}(\varphi) \subset G/\Gamma$ its compact support. Choose a relatively compact open neighborhood $V \subset G$ of $e \in G$. Then there exist $g_1, \dots, g_n \in G$ such that $Q \subset \bigcup_{i=1}^n p(g_i V)$. Set $K = p^{-1}(Q) \cap \bigcup_{i=1}^n g_i \overline{V}$. Then $K \subset G$ is a compact subset such that $p(K) = Q$. By Urysohn's lemma (see e.g. [**DE14**, Lemma A.8.1(ii)]), we may choose $f_K \in \mathrm{C}_c(G)_+$ such that $f|_K \equiv 1_K$.

Define the function $f : G \to \mathbb{C}$ by the formula $f(g) = \frac{\varphi(g\Gamma)}{\mathscr{T}(f_K)(g\Gamma)} f_K(g)$ if $\mathscr{T}(f_K)(g\Gamma) \neq 0$ and $f(g) = 0$ otherwise. Then $\mathrm{supp}(f) \subset \mathrm{supp}(f_K)$ is compact and f is continuous on G since $\mathscr{T}(f_K)(g\Gamma) > 0$ on a neighborhood of Q. Thus, $f \in \mathrm{C}_c(G)$ and we have $\mathscr{T}(f) = \varphi$.

Proposition 1.14. *Let G be a locally compact group and $\Gamma < G$ a uniform discrete subgroup. Then G is unimodular and $\Gamma < G$ is a lattice.*

If G is moreover compactly generated, then $\Gamma < G$ is finitely generated.

PROOF. Fix a right invariant Haar measure μ_G on G. Consider the positive linear functional

$$\Phi : \mathrm{C}_c(G/\Gamma) \to \mathbb{C} : \overline{f} \mapsto \int_G f(g)\, \mathrm{d}\mu_G(g).$$

In order to check that Φ is well-defined, it suffices to show that if $\varphi \in \mathrm{C}_c(G)$ is such that $\overline{\varphi} = 0$, then we have $\int_G \varphi(g)\, \mathrm{d}\mu_G(g) = 0$. Indeed, for every $\psi \in \mathrm{C}_c(G)$, using Fubini's theorem, we have

$$\begin{aligned}\int_G \overline{\varphi}(h\Gamma)\psi(h)\, \mathrm{d}\mu_G(h) &= \sum_{\gamma\in\Gamma} \int_G \varphi(h\gamma)\psi(h)\, \mathrm{d}\mu_G(h) \\ &= \sum_{\gamma\in\Gamma} \int_G \varphi(h)\psi(h\gamma^{-1})\, \mathrm{d}\mu_G(h) \\ &= \int_G \varphi(h)\overline{\psi}(h\Gamma)\, \mathrm{d}\mu_G(h).\end{aligned}$$

Since the map $\mathrm{C}_c(G) \to \mathrm{C}_c(G/\Gamma) : f \mapsto \overline{f}$ is surjective, there exists $\psi \in \mathrm{C}_c(G)$ such that $\overline{\psi} \equiv 1$ on the compact subset $\mathrm{supp}(\varphi)\Gamma \subset G/\Gamma$. Therefore, we obtain

$$\int_G \varphi(h)\, \mathrm{d}\mu_G(h) = \int_G \varphi(h)\overline{\psi}(h\Gamma)\, \mathrm{d}\mu_G(h) = \int_G \overline{\varphi}(h\Gamma)\psi(h)\, \mathrm{d}\mu_G(h) = 0.$$

By Riesz's representation theorem, there exists a unique regular Borel measure ν on G/Γ such that

$$\forall f \in \mathrm{C}_c(G), \quad \int_G f(h)\, \mathrm{d}\mu_G(h) = \int_G \overline{f}(h\Gamma)\, \mathrm{d}\nu(h\Gamma).$$

Note that the above argument does not use the fact that $\Gamma < G$ is uniform.

However, since $\Gamma < G$ is uniform, G/Γ is compact and we have $0 < \nu(G/\Gamma) < +\infty$. Up to normalization, we may assume that $\nu(G/\Gamma) = 1$.

Define the left invariant Haar measure m_G on G by the formula $m_G(B) = \mu_G(B^{-1})$ for every $B \in \mathscr{B}(G)$. Then for every $B \in \mathscr{B}(G)$ and every $g \in G$, we have

$$(g_*\mu_G)(B) = \mu_G(g^{-1}B) = m_G(B^{-1}g) = \Delta_G(g)\, m_G(B^{-1}) = \Delta_G(g)\, \mu_G(B)$$

and so $g_*\mu_G = \Delta_G(g)\, \mu_G$. By uniqueness in the previous construction, we obtain $g_*\nu = \Delta_G(g)\, \nu$ for every $g \in G$. Since $\nu \in \mathrm{Prob}(G/\Gamma)$ is a probability measure, we obtain $\Delta_G(g) = 1$ and $g_*\nu = \nu$ for every $g \in G$. Thus, $\Delta_G \equiv 1$ and so G is unimodular. Moreover, $\nu \in \mathrm{Prob}(G/\Gamma)$ is G-invariant and so $\Gamma < G$ is a lattice.

Assume moreover that G is compactly generated. Choose a compact subset $Q \subset G$ such that $e \in Q$ and $G = \bigcup_{n\geq 1} Q^n$. Since G/Γ is compact, we may choose a compact subset $K \subset G$ such that $p(K) = G/\Gamma$ (see the proof of surjectivity of the map $\mathscr{T} : \mathrm{C}_c(G) \to \mathrm{C}_c(G/\Gamma)$). Upon replacing Q by $Q \cup K$, we may further assume that $Q \cdot \Gamma = G$. Then $S_0 = Q \cap \Gamma$ is finite. Moreover, since Q^2 is compact, there exists a finite subset $S_1 \subset \Gamma$

such that $Q^2 \subset QS_1$. Indeed, otherwise we could find sequences $(g_n)_n$ in Q^2, $(h_n)_n$ in Q and $(\gamma_n)_n$ in Γ such that $g_n = h_n\gamma_n$ for every $n \in \mathbb{N}$ and $(\gamma_n)_n$ are pairwise distinct. This would imply that $\gamma_n = h_n^{-1}g_n \in Q^3 \cap \Gamma$ for every $n \in \mathbb{N}$. Since Q^3 is compact and $\Gamma < G$ is discrete, $Q^3 \cap \Gamma$ must be finite, a contradiction. Set $S = S_0 \cup S_1 \subset \Gamma$. Then $Q \cap \Gamma \subset S$ and for every $n \geq 1$, we have $Q^{n+1} \subset QS^n$. We claim that S is a finite generating set for Γ. Indeed, by construction, we have $Q \cap \Gamma \subset S$. Next, let $n \geq 1$ and $\gamma \in Q^{n+1} \cap \Gamma \subset QS^n \cap \Gamma$. Then $\gamma = g\gamma_n$ where $g \in Q$ and $\gamma_n \in S^n$. This implies that $\gamma\gamma_n^{-1} = g \in Q \cap \Gamma \subset S$. Then $\gamma = g\gamma_n \in SS^n = S^{n+1}$ and hence $Q^{n+1} \cap \Gamma \subset S^{n+1}$. This implies that $\Gamma = \bigcup_{n\geq 1} Q^n \cap \Gamma \subset \bigcup_{n \geq 1} S^n$ and so Γ is finitely generated. □

PROPOSITION 1.15. *Let G be a locally compact group that possesses a lattice $\Gamma < G$. Then G is unimodular. Moreover, there is a unique G-invariant regular Borel probability measure $\nu \in \mathrm{Prob}(G/\Gamma)$.*

PROOF. Let $\nu \in \mathrm{Prob}(G/\Gamma)$ be a G-invariant regular Borel probability measure. We claim that there exists a unique left invariant Haar measure m_G on G such that

$$\forall f \in \mathrm{C}_c(G), \quad \int_G f(h)\,\mathrm{d}m_G(h) = \int_{G/\Gamma} \overline{f}(g\Gamma)\,\mathrm{d}\nu(g\Gamma). \tag{1.3}$$

Indeed, the well-defined positive linear functional

$$\mathrm{C}_c(G) \to \mathbb{C} : f \mapsto \int_{G/\Gamma} \overline{f}(g\Gamma)\,\mathrm{d}\nu(g\Gamma)$$

is left invariant. By Riesz's representation theorem, there exists a unique left invariant Haar measure m_G on G for which (1.3) holds.

Applying (1.1), for every $f \in \mathrm{C}_c(G)$ and every $\gamma \in \Gamma$, letting $f_\gamma = f(\,\cdot\,\gamma^{-1}) \in \mathrm{C}_c(G)$, we have

$$\begin{aligned}\Delta_G(\gamma)\int_G f(h)\,\mathrm{d}m_G(h) &= \int_G f_\gamma(h)\,\mathrm{d}m_G(h)\\ &= \int_{G/\Gamma} \overline{f_\gamma}(h\Gamma)\,\mathrm{d}\nu(h\Gamma)\\ &= \int_{G/\Gamma} \overline{f}(h\Gamma)\,\mathrm{d}\nu(h\Gamma)\\ &= \int_G f(h)\,\mathrm{d}m_G(h).\end{aligned}$$

This implies that $\Delta_G(\gamma) = 1$ for every $\gamma \in \Gamma$. Consider the well-defined continuous mapping $\overline{\Delta} : G/\Gamma \to \mathbb{R}_+^* : g\Gamma \mapsto \Delta_G(g)$. Then $\eta = \overline{\Delta}_*\nu \in \mathrm{Prob}(\mathbb{R}_+^*)$ is a Borel probability measure that is invariant under multiplication by $\Delta_G(g)$ for every $g \in G$. This implies that $\Delta_G \equiv 1$ and so G is unimodular.

Observe that (1.3) together with surjectivity of $\mathscr{T} : \mathrm{C}_c(G) \to \mathrm{C}_c(G/\Gamma)$ imply that there is a unique G-invariant regular Borel probability measure $\nu \in \mathrm{Prob}(G/\Gamma)$. □

The next proposition provides a group-theoretic characterization of uniform lattices in locally compact groups.

PROPOSITION 1.16. *Let G be a locally compact group and $\Gamma < G$ a lattice. The following assertions are equivalent:*

(i) *$\Gamma < G$ is uniform.*
(ii) *There exists a compact neighborhood $U \subset G$ of $e \in G$ such that for every $g \in G$, we have $g\Gamma g^{-1} \cap U = \{e\}$.*

PROOF. (i) ⇒ (ii) Assume that $\Gamma < G$ is uniform. Since $\Gamma < G$ is discrete, we may choose a compact neighborhood $W \subset G$ of $e \in G$ such that $\Gamma \cap W = \{e\}$. Next, we may choose a symmetric compact neighborhood $V \subset W$ of $e \in G$ such that $VVV \subset W$. Observe that for every $h \in V$, we have

$$h\Gamma h^{-1} \cap V \subset h(\Gamma \cap h^{-1}Vh)h^{-1} \subset h(\Gamma \cap W)h^{-1} = \{e\}.$$

By compactness of G/Γ, there exist $n \geq 1$ and $g_1, \dots, g_n \in G$ such that $G/\Gamma = \bigcup_{i=1}^n g_i p(V)$. Set $U = \bigcap_{i=1}^n g_i V g_i^{-1}$. Then for every $g \in G$, there exist $1 \leq i \leq n$ and $h \in V$ such that $g\Gamma = g_i h\Gamma$ and hence

$$g\Gamma g^{-1} \cap U = g_i h\Gamma h^{-1} g_i^{-1} \cap U \subset g_i(h\Gamma h^{-1} \cap V)g_i^{-1} = \{e\}.$$

(ii) ⇒ (i) Denote by $\nu \in \mathrm{Prob}(G/\Gamma)$ the unique G-invariant regular Borel probability measure and by m_G the unique Haar measure on G such that (1.3) holds. Assume that there exists such a compact neighborhood $U \subset G$ of $e \in G$. Choose a compact neighborhood $V \subset G$ of $e \in G$ such that $V^{-1}V \subset U$. Choose a nonnegative function $\varphi \in \mathrm{C}_c(G)$ such that $0 \leq \varphi \leq 1$ and $\mathrm{supp}(\varphi) \subset V$. Set $\varepsilon = \int_G \varphi(h)\, \mathrm{d}m_G(h)$.

For every $g \in G$, define $\varphi_g = \varphi(\,\cdot\, g^{-1}) \in \mathrm{C}_c(G)$. Note that $0 \leq \varphi_g \leq 1$ and $\mathrm{supp}(\varphi_g) \subset Vg$. Moreover, we have $\mathrm{supp}(\overline{\varphi_g}) \subset Vg\Gamma$. Since m_G is right invariant, we have

$$\begin{aligned}
\varepsilon &= \int_G \varphi(h)\, \mathrm{d}m_G(h) \\
&= \int_G \varphi_g(h)\, \mathrm{d}m_G(h) \\
&= \int_{G/\Gamma} \overline{\varphi_g}(h\Gamma)\, \mathrm{d}\nu(h\Gamma) \\
&= \int_{Vg\Gamma} \overline{\varphi_g}(h\Gamma)\, \mathrm{d}\nu(h\Gamma) \\
&= \int_{Vg\Gamma} \sum_{\gamma \in \Gamma} \varphi_g(h\gamma)\, \mathrm{d}\nu(h\Gamma).
\end{aligned}$$

We claim that for every $h \in Vg\Gamma$, there is at most one $\gamma \in \Gamma$ such that $h\gamma \in Vg$. Indeed, if $\gamma_1, \gamma_2 \in \Gamma$ are elements such that $h\gamma_1, h\gamma_2 \in Vg$, then $g\gamma_1^{-1}\gamma_2 g^{-1} \in V^{-1}V \subset U$. Since $g\Gamma g^{-1} \cap U = \{e\}$, we have $\gamma_1 = \gamma_2$. Since $0 \leq \varphi_g \leq 1$ and $\mathrm{supp}(\varphi_g) \subset V_g$, it follows that

$$\varepsilon = \int_{Vg\Gamma} \sum_{\gamma \in \Gamma} \varphi_g(h\gamma)\, \mathrm{d}\nu(h\Gamma) \leq \int_{Vg\Gamma} 1\, \mathrm{d}\nu(h\Gamma) = \nu(Vg\Gamma).$$

We have showed that $\nu(Vg\Gamma) \geq \varepsilon$ for every $g \in G$.

Let $F \subset G$ be a finite subset for which for every $g, h \in F$ such that $g \neq h$, we have $Vg\Gamma \cap Vh\Gamma = \emptyset$. Then we have

$$\sharp F \cdot \varepsilon \leq \sum_{g \in F} \nu(Vg\Gamma) = \nu(\bigcup_{g \in F} Vg\Gamma) \leq 1$$

and hence $\sharp F \leq \varepsilon^{-1}$. We may then choose a maximal finite subset $F \subset G$ with the aforementioned property. It follows that for every $g \in G$, we have $Vg\Gamma \cap VF\Gamma \neq \emptyset$ and hence $g\Gamma \in V^{-1}VF\Gamma \subset UF\Gamma$. Since $UF\Gamma \subset G/\Gamma$ is compact, it follows that $G/\Gamma = UF\Gamma$ is compact. □

When G is a locally compact second countable group, we prove a very useful criterion to ensure that a discrete subgroup $\Gamma < G$ is a lattice.

THEOREM 1.17. *Let G be a locally compact second countable group and $\Gamma < G$ a discrete subgroup. The following assertions are equivalent:*

- (i) *$\Gamma < G$ is a lattice.*
- (ii) *G is unimodular and there is a Borel fundamental domain $\mathscr{F} \subset G$ for the right translation action $\Gamma \curvearrowright G$ such that $0 < m_G(\mathscr{F}) < +\infty$.*
- (iii) *G is unimodular and there is a Borel subset $\mathfrak{S} \subset G$ such that $\mathfrak{S} \cdot \Gamma = G$ and $0 < m_G(\mathfrak{S}) < +\infty$.*

PROOF. Recall that since G is a locally compact second countable group, the discrete subgroup $\Gamma < G$ is necessarily countable.

(i) ⇒ (ii) We already know that G is unimodular by Proposition 1.15. Denote by $\nu \in \mathrm{Prob}(G/\Gamma)$ the unique G-invariant regular Borel probability measure. Denote by m_G the unique Haar measure on G satisfying (1.3). Since G is locally compact second countable, (1.3) holds for every nonnegative Borel function $f : G \to \mathbb{R}_+$. In particular, for $f = \mathbf{1}_{\mathscr{F}}$, we have $\overline{f} \equiv 1$ and so

$$m_G(\mathscr{F}) = \int_G f(h)\, \mathrm{d}m_G(h) = \int_{G/\Gamma} \overline{f}\, \mathrm{d}\nu(h\Gamma) = 1 < +\infty.$$

Since $m_G(G) > 0$, $G = \bigcup_{\gamma \in \Gamma} \mathscr{F}\gamma$ and $m_G(\mathscr{F}\gamma) = m_G(\mathscr{F})$ for every $\gamma \in \Gamma$, we also have $m_G(\mathscr{F}) > 0$.

(ii) ⇒ (iii) It is trivial.

(iii) $\Rightarrow$ (i) Following the proof of Proposition 1.14 and since m_G is right invariant, we may consider the well-defined nonzero left invariant linear functional

$$\Phi : \mathrm{C}_c(G/\Gamma) \to \mathbb{C} : \overline{f} \mapsto \int_G f(g)\, \mathrm{d}m_G(g).$$

By Riesz's representation theorem, there exists a unique nonzero G-invariant regular Borel measure ν on G/Γ such that (1.3) holds. Since G is locally compact second countable, (1.3) holds for every nonnegative Borel function $f : G \to \mathbb{R}_+$. In particular, for $f = \mathbf{1}_{\mathfrak{S}}$, we have $\overline{f} \geq 1$ and so

$$\nu(G/\Gamma) \leq \int_{G/\Gamma} \overline{f}\, \mathrm{d}\nu(h\Gamma) = \int_G f(h)\, \mathrm{d}m_G(h) = m_G(\mathfrak{S}) < +\infty.$$

Then $\frac{1}{\nu(G/\Gamma)}\nu \in \mathrm{Prob}(G/\Gamma)$ is a G-invariant regular Borel probability measure and so $\Gamma < G$ is a lattice. $\square$

Let us point out that when $\Gamma < G$ is a lattice, all Borel fundamental domains for the right translation action $\Gamma \curvearrowright G$ have the same finite Haar measure. Indeed, whenever $\mathscr{F}_1, \mathscr{F}_2 \subset G$ are Borel fundamental domains, since the Haar measure m_G on G is right invariant, we have

$$\begin{aligned} m_G(\mathscr{F}_1) &= \sum_{\gamma \in \Gamma} m_G(\mathscr{F}_1 \cap \mathscr{F}_2 \gamma) \\ &= \sum_{\gamma \in \Gamma} m_G(\mathscr{F}_1 \gamma^{-1} \cap \mathscr{F}_2) \\ &= m_G(\mathscr{F}_2). \end{aligned}$$

Examples 1.18. Here are some examples of lattices in locally compact groups.

(i) For every $d \geq 1$, the discrete subgroup $\mathbb{Z}^d < \mathbb{R}^d$ is a uniform lattice.
(ii) More generally, any lattice $\Gamma < G$ in a locally compact second countable abelian group G is necessarily uniform.
(iii) The discrete Heisenberg group $\mathrm{H}_3(\mathbb{Z}) < \mathrm{H}_3(\mathbb{R})$ is a uniform lattice in the continuous Heisenberg group $\mathrm{H}_3(\mathbb{R})$:

$$\mathrm{H}_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$$
$$\mathrm{H}_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

(iv) More generally, any lattice $\Gamma < G$ in a locally compact second countable nilpotent group G is necessarily uniform.

3. $\mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SL}_d(\mathbb{R})$, $d \geq 2$

In this section, we prove the following theorem due to Minkowski.

THEOREM 1.19 (Minkowski). *For every $d \geq 2$, the discrete subgroup $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is a nonuniform lattice.*

Before proving Theorem 1.19, we need to prove some preliminary results that are also of independent interest.

Let $d \geq 1$. Endow $\mathbb{R}^d$ with its canonical euclidean structure. Denote by $K = \mathrm{SO}_d(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$ the special orthogonal subgroup and observe that $K < \mathrm{SL}_d(\mathbb{R})$ is compact. Denote by $A < \mathrm{SL}_d(\mathbb{R})$ the subgroup of diagonal matrices with positive entries, that is,

$$A = \{a = \mathrm{diag}(\lambda_1, \dots, \lambda_d) \mid \lambda_1, \dots, \lambda_d > 0,\ \lambda_1 \cdots \lambda_d = 1\} < \mathrm{SL}_d(\mathbb{R}).$$

Denote by $N = \mathrm{T}_d(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$ the strict upper triangular subgroup as in Example 1.10(vi).

LEMMA 1.20 (Iwasawa decomposition). *The map $K \times A \times N \to \mathrm{SL}_d(\mathbb{R}) : (k, a, n) \mapsto kan$ is a homeomorphism. We simply write $\mathrm{SL}_d(\mathbb{R}) = K \cdot A \cdot N$.*

PROOF. Denote by $(e_1, \dots, e_d)$ the canonical basis of $\mathbb{R}^d$. The map $\Psi : K \times A \times N \to \mathrm{SL}_d(\mathbb{R}) : (k, a, n) \mapsto kan$ is clearly continuous. Conversely, let $g \in \mathrm{SL}_d(\mathbb{R})$ and write $v_i = ge_i \in \mathbb{R}^d$ for every $1 \leq i \leq d$. By Gram–Schmidt's orthogonalization process, set $w_1 = v_1$ and $w_{i+1} = v_{i+1} - P_{V_i}(v_{i+1})$ where $V_i = \mathrm{Vect}(v_1, \dots, v_i)$ for every $1 \leq i \leq d-1$. Then $(\frac{w_1}{\|w_1\|}, \dots, \frac{w_d}{\|w_d\|})$ is an orthonormal basis for $\mathbb{R}^d$ and we may find $k \in \mathrm{O}_d(\mathbb{R})$ such that $ke_i = \frac{w_i}{\|w_i\|}$ for every $1 \leq i \leq d$. Then the matrix $k^{-1}g$ is upper triangular and $(k^{-1}g)_{ii} = \|w_i\|$ for every $1 \leq i \leq d$. It follows that $\det(k^{-1}) = \det(k^{-1}g) = \|w_1\| \cdots \|w_d\| > 0$ and hence $k \in \mathrm{SO}_d(\mathbb{R})$. Letting $a = \mathrm{diag}(\|w_1\|, \dots, \|w_d\|) \in A$, we have $g = kan$ and the map $\mathrm{SL}_d(\mathbb{R}) \to K \times A \times N : g \mapsto (k, a, n)$ is continuous. Since its inverse is Ψ, we have showed that $\Psi : K \times A \times N \to \mathrm{SL}_d(\mathbb{R}) : (k, a, n) \mapsto kan$ is a homeomorphism. □

LEMMA 1.21. *Endow $(K, \mathrm{d}k)$, $(A, \mathrm{d}a)$, $(N, \mathrm{d}n)$ with their respective Haar measure. Then the pushforward measure of*

$$\prod_{1 \leq i < j \leq d} \frac{\lambda_i}{\lambda_j} \, \mathrm{d}k \, \mathrm{d}a \, \mathrm{d}n$$

under the map $K \times A \times N \to \mathrm{SL}_d(\mathbb{R}) : (k, a, n) \mapsto kan$ is a Haar measure on $\mathrm{SL}_d(\mathbb{R})$.

PROOF. Consider the product map $\Psi : K \times AN \to \mathrm{SL}_d(\mathbb{R}) : (k, p) \mapsto k^{-1}p$. Since $\mathrm{SL}_d(\mathbb{R})$ is unimodular, the regular Borel measure $(\Psi^{-1})_* m_{\mathrm{SL}_d(\mathbb{R})}$ on $K \times AN$ is right invariant. Then $(\Psi^{-1})_* m_{\mathrm{SL}_d(\mathbb{R})}$ is a right invariant Haar measure on the locally compact second countable group $K \times AN$ and hence $(\Psi^{-1})_* m_{\mathrm{SL}_d(\mathbb{R})} = \mu_K \otimes \mu_{AN}$ where μ_K is a right invariant Haar measure on

K and μ_{AN} is a right invariant Haar measure on AN. Since K is compact, μ_K is also left invariant and hence we may assume that $\mathrm{d}\mu_K(k) = \mathrm{d}k$. It remains to prove that $\prod_{1\leq i<j\leq d} \frac{\lambda_i}{\lambda_j} \mathrm{d}a\,\mathrm{d}n$ is a right invariant Haar measure on AN.

As explained in Examples 1.10(vi), we may assume that $\mathrm{d}m_N(n) = \mathrm{d}n = \prod_{1\leq i<j\leq d} \mathrm{d}n_{ij}$. Observe that $N \lhd AN$ is a normal subgroup and define the conjugation action $\mathrm{Ad} : A \curvearrowright N$ by $\mathrm{Ad}(a)(n) = ana^{-1}$ for $a \in A$, $n \in N$. Then $AN = A \ltimes N$ and $\mathrm{d}a\,\mathrm{d}n$ is a left invariant measure on AN by Proposition 1.9. A simple calculation shows that $\mathrm{Ad}(a)_* m_N = (\prod_{1\leq i<j\leq d} \frac{\lambda_i}{\lambda_j})^{-1} \cdot m_N$. Then Proposition 1.9 implies that $\prod_{1\leq i<j\leq d} \frac{\lambda_i}{\lambda_j} \mathrm{d}a\,\mathrm{d}n$ is a right invariant Haar measure on AN. □

For all $t, u > 0$, set

$$\begin{aligned} A_t &= \{a = \mathrm{diag}(\lambda_1, \ldots, \lambda_d) \in A \mid \forall 1 \leq i \leq d-1, \lambda_i \leq t\lambda_{i+1}\} \\ N_u &= \{n = (n_{ij})_{ij} \in N \mid \forall 1 \leq i < j \leq d, |n_{ij}| \leq u\} \\ \mathfrak{S}_{t,u} &= K \cdot A_t \cdot N_u. \end{aligned}$$

The Borel subset $\mathfrak{S}_{t,u} \subset G$ is called a *Siegel domain*. We now have all the tools to prove Theorem 1.19.

Proof of Theorem 1.19. For every $t \geq \frac{2}{\sqrt{3}}$ and every $u \geq \frac{1}{2}$, we show that $\mathrm{SL}_d(\mathbb{R}) = \mathfrak{S}_{t,u} \cdot \mathrm{SL}_d(\mathbb{Z})$ and that $\mathfrak{S}_{t,u}$ has finite Haar measure. By Theorem 1.17, this implies that $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is a lattice. We divide the proof into a series of claims.

Claim 1.22. For all $t, u > 0$, the Siegel domain $\mathfrak{S}_{t,u}$ has finite Haar measure.

Indeed, note that since K and N_u are both compact in $\mathrm{SL}_d(\mathbb{R})$, using Lemma 1.21 it suffices to prove that

$$\kappa_t = \int_{A_t} \prod_{1\leq i<j\leq d} \frac{\lambda_i}{\lambda_j} \, \mathrm{d}a < +\infty.$$

Observe that the map

$$\Theta : A \to \mathbb{R}^{d-1} : \mathrm{diag}(\lambda_1, \ldots, \lambda_d) \mapsto \left(\log \frac{\lambda_2}{\lambda_1}, \ldots, \log \frac{\lambda_d}{\lambda_{d-1}}\right)$$

is a topological group isomorphism. We may choose the Haar measure $\mathrm{d}a$ on A that is the pushforward of the Lebesgue measure on $\mathbb{R}^{d-1}$ by Θ^{-1}. We then have

$$\begin{aligned} \kappa_t &= \int_{\mathbb{R}^{d-1}} \prod_{1\leq i<j\leq d} \exp(-(s_i + \cdots + s_{j-1})) \mathbf{1}_{\{s_1,\ldots,s_{d-1} \geq -\log t\}} \, \mathrm{d}s_1 \cdots \mathrm{d}s_{d-1} \\ &= \prod_{k=1}^{d-1} \int_{-\log t}^{+\infty} \exp(-k(d-k)s_k) \, \mathrm{d}s_k < +\infty. \end{aligned}$$

CLAIM 1.23. For every $u \geq \frac{1}{2}$, we have $N = N_u \cdot (N \cap \mathrm{SL}_d(\mathbb{Z}))$.

Indeed, it suffices to prove Claim 1.23 for $u = \frac{1}{2}$. We proceed by induction over $d \geq 1$. For $d = 1$, there is nothing to prove. Assume that the result is true for $d-1 \geq 1$ and let us prove it for d. Let $n \in N = \mathrm{T}_d(\mathbb{R})$ that we write

$$n = \begin{pmatrix} 1 & * \\ 0 & n_0 \end{pmatrix} \quad \text{where} \quad n_0 \in \mathrm{T}_{d-1}(\mathbb{R}).$$

By induction hypothesis, there exists $\gamma_0 \in \mathrm{T}_{d-1}(\mathbb{R}) \cap \mathrm{SL}_{d-1}(\mathbb{Z})$ such that $n_1 = n_0\gamma_0^{-1} \in \mathrm{T}_{d-1}(\mathbb{R})_{1/2}$. Write

$$n \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0^{-1} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & n_1 \end{pmatrix} \quad \text{where} \quad x \in \mathbb{R}^{d-1}.$$

Choose $y \in \mathbb{Z}^{d-1}$ such that $x - y \in [-1/2, 1/2]^{d-1}$. Then

$$\begin{aligned} n &= \begin{pmatrix} 1 & x \\ 0 & n_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x-y \\ 0 & n_1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0 \end{pmatrix} \end{aligned}$$

where

$$\begin{pmatrix} 1 & x-y \\ 0 & n_1 \end{pmatrix} \in N_{1/2} \quad \text{and} \quad \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0 \end{pmatrix} \in N \cap \mathrm{SL}_d(\mathbb{Z}).$$

This shows the result is true for d and finishes the proof of Claim 1.23.

CLAIM 1.24. For every $t \geq \frac{2}{\sqrt{3}}$, we have $\mathrm{SL}_d(\mathbb{R}) = K \cdot A_t \cdot N \cdot \mathrm{SL}_d(\mathbb{Z})$.

Indeed, it suffices to prove Claim 1.24 for $t = \frac{2}{\sqrt{3}}$. We proceed by induction over $d \geq 1$. For $d = 1$, there is nothing to prove. Assume that the result is true for $d-1 \geq 1$ and let us prove it for d. Denote by $(e_1, \ldots, e_d)$ the canonical basis of $\mathbb{R}^d$. Let $g \in \mathrm{SL}_d(\mathbb{R})$. Since $\Lambda = g\mathbb{Z}^d$ is a lattice in $\mathbb{R}^d$, there must exist a vector $v_1 \in \Lambda \setminus \{0\}$ such that

$$\|v_1\| = \min \{\|v\| \mid v \in \Lambda \setminus \{0\}\}.$$

By minimality of the norm of $v_1 \in \Lambda \setminus \{0\}$, we may find $v_2, \ldots, v_d \in \Lambda \setminus \{0\}$ such that $(v_1, \ldots, v_d)$ is a basis of Λ (see e.g. [**Ca71**, Corollary I.3]). Upon further replacing v_1 by $-v_1$, there exists $\gamma \in \mathrm{SL}_d(\mathbb{Z})$ such that $\gamma e_i = g^{-1}v_i$ for every $1 \leq i \leq d$. Note that $g\gamma e_1 = v_1$.

Next, consider the Iwasawa decomposition $g\gamma = kan$ and write

$$an = \begin{pmatrix} \lambda^{d-1} & * \\ 0 & \lambda^{-1} g_0 \end{pmatrix} \quad \text{where} \quad \lambda \in \mathbb{R}_+^*,\, g_0 \in \mathrm{SL}_{d-1}(\mathbb{R}).$$

By induction hypothesis, there exist $k_0 \in \mathrm{SO}_{d-1}(\mathbb{R})$ and $\gamma_0 \in \mathrm{SL}_{d-1}(\mathbb{Z})$ such that $k_0^{-1} g_0 \gamma_0^{-1} \in (A_{d-1})_{2/\sqrt{3}} \cdot \mathrm{T}_{d-1}(\mathbb{R})$. If we consider

$$h = \begin{pmatrix} 1 & 0 \\ 0 & k_0^{-1} \end{pmatrix} k^{-1} g\gamma \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{d-1} & * \\ 0 & \lambda^{-1}\, k_0^{-1} g_0 \gamma_0^{-1} \end{pmatrix} \in AN$$

we obtain that the diagonal coefficients of h satisfy $h_{i,i} \leq \frac{2}{\sqrt{3}} h_{i+1,i+1}$ for every $2 \leq i \leq d-1$. It remains to prove that $h_{1,1} \leq \frac{2}{\sqrt{3}} h_{2,2}$. Observe that for every $w \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\|he_1\| = \|g\gamma \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0^{-1} \end{pmatrix} e_1\| = \|g\gamma e_1\| = \|v_1\| \leq \|g\gamma \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0^{-1} \end{pmatrix} w\| = \|hw\|.$$

Using Claim 1.23, write $h = \mathrm{diag}(h_{11}, \dots, h_{dd}) n_1 \gamma_1$ where $n_1 \in N_{1/2}$ and $\gamma_1 \in N \cap \mathrm{SL}_d(\mathbb{Z})$. Then $he_1 = \mathrm{diag}(h_{11}, \dots, h_{dd}) e_1 = h_{11} e_1$ and with $w = \gamma_1^{-1} e_2 \in \mathbb{Z}^d \setminus \{0\}$, we have $hw = \mathrm{diag}(h_{11}, \dots, h_{dd}) n_1 e_2 = h_{11} n_{12} e_1 + h_{22} e_2$. Then we obtain

$$h_{11}^2 = \|he_1\|^2 \leq \|hw\|^2 = h_{11}^2 n_{12}^2 + h_{22}^2 \leq \frac{1}{4} h_{11}^2 + h_{22}^2$$

and so $h_{11}^2 \leq \frac{4}{3} h_{22}^2$. This finishes the proof of Claim 1.24.

A combination of Claims 1.22, 1.23, 1.24 and Theorem 1.17 implies that $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is a lattice.

It remains to prove that $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is nonuniform. Indeed, regard $\mathrm{SL}_2(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$ as a subgroup in the top left corner and set

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{Z}).$$

Then a simple calculation shows that

$$g_n \gamma g_n^{-1} = \begin{pmatrix} 1 & n^{-2} \\ 0 & 1 \end{pmatrix} \to e \quad \text{with} \quad g_n = \begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R}).$$

Then Proposition 1.16 implies that $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is nonuniform. □

Let $r \geq 2$ and $G_1, \dots, G_r$ be locally compact groups. Set $G = \prod_{i=1}^r G_i$. For every $1 \leq i \leq r$, set $\widehat{G}_i = \prod_{j \neq i} G_j$ and denote by $p_i : G \to \widehat{G}_i$ the canonical factor map.

Definition 1.25. Let $\Gamma < G$ be a discrete subgroup. We say that $\Gamma < G$ is *irreducible* if for every $1 \leq i \leq r$, the image $p_i(\Gamma)$ is dense in $\widehat{G}_i$.

Example 1.26. Here are some examples of discrete irreducible subgroups $\Gamma < G$ in locally compact groups.

(i) Let $q \geq 2$ be a square-free integer. Define the field automorphism $\sigma : \mathbb{Q}(\sqrt{q}) \to \mathbb{Q}(\sqrt{q}) : x + y\sqrt{q} \mapsto x - y\sqrt{q}$. For every $d \geq 2$, the subgroup

$$\Gamma = \{(g, g^\sigma) \mid g \in \mathrm{SL}_d(\mathbb{Z}[\sqrt{q}])\} < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R})$$

is discrete and irreducible. Write $\mathrm{SL}_d(\mathbb{Z}[\sqrt{q}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R})$.

(ii) Let $p \in \mathscr{P}$ be a prime. For every $d \geq 2$, the subgroup

$$\Gamma = \left\{(g, g) \mid g \in \mathrm{SL}_d(\mathbb{Z}[p^{-1}])\right\} < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{Q}_p)$$

is discrete and irreducible. Write $\mathrm{SL}_d(\mathbb{Z}[p^{-1}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{Q}_p)$.

Borel–Harish-Chandra's results [**BHC61**] provide many examples of irreducible lattices in locally compact groups. We refer the reader to [**Ma91**, Chapter IX] and [**Be09**, §2] for further details.

Examples 1.27. Let $d \geq 2$.

(i) The discrete subgroup $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$ is a nonuniform lattice (see Theorem 1.19).

(ii) For every square-free integer $q \geq 2$, the discrete subgroup
$$\mathrm{SL}_d(\mathbb{Z}[\sqrt{q}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R})$$
is a nonuniform irreducible lattice.

(iii) For every prime $p \in \mathscr{P}$, the discrete subgroup
$$\mathrm{SL}_d(\mathbb{Z}[p^{-1}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{Q}_p)$$
is a nonuniform irreducible lattice.

(iv) More generally, for every finite set of primes $S = \{p_1, \ldots, p_r\} \subset \mathscr{P}$, the discrete subgroup
$$\mathrm{SL}_d(\mathbb{Z}[S^{-1}]) < \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{SL}_d(\mathbb{Q}_{p_r})$$
is a nonuniform irreducible lattice.

(v) Let $d \geq 3$ and $p \geq q \geq 1$ such that $p + q = d$. Define
$$J_{p,q} = \begin{pmatrix} 1_p & 0 \\ 0 & -\sqrt{2}\,1_q \end{pmatrix}$$
$$\Gamma = \left\{ g \in \mathrm{SL}_d(\mathbb{Z}[\sqrt{2}]) \mid g J_{p,q} {}^t g = J_{p,q} \right\}$$
$$G = \left\{ g \in \mathrm{SL}_d(\mathbb{R}) \mid g J_{p,q} {}^t g = J_{p,q} \right\}.$$
Then $\Gamma < G$ is a uniform lattice.

CHAPTER 2

Ergodic group theory

In this chapter, we give an introduction to ergodic group theory. We prove a dynamical dichotomy result for continuous isometric actions of $\mathrm{SL}_d(\mathbb{R})$, $d \geq 2$. We also discuss unitary representation theory for locally compact groups in relation with ergodic theory. Finally, we investigate the notion of amenability for groups and group actions.

1. Ergodic theory

1.1. Topological dynamics. In this subsection, we give an introduction to topological dynamics and we prove a dynamical dichotomy result for continuous isometric actions of $\mathrm{SL}_d(\mathbb{R})$, $d \geq 2$.

Let G be a locally compact group, X a Hausdorff topological space and $G \curvearrowright X$ a *continuous* action in the sense that the action map $G \times X \to X : (g, x) \mapsto gx$ is continuous. For every $x \in X$, we denote by $Gx = \{gx \mid g \in G\} \subset X$ its G-orbit and by $\mathrm{Stab}_G(x) = \{g \in G \mid gx = x\} < G$ its stabilizer subgroup (note that $\mathrm{Stab}_G(x) < G$ is closed subgroup). Denote by $G\backslash X = \{Gx \mid x \in X\}$ the quotient space endowed with the quotient topology. Then the quotient map $p : X \to G\backslash X : x \mapsto Gx$ is continuous and open. In general, $G\backslash X$ behaves pathologically with respect to the quotient topology.

Firstly, we investigate when the quotient space $G\backslash X$ is T_0. A topological space Z is said to be T_0 if for all $z_1, z_2 \in Z$ such that $z_1 \neq z_2$, there exists an open set $U \subset Z$ such that $z_1 \in U$ and $z_2 \notin U$ or $z_1 \notin U$ and $z_2 \in U$. A subset $Y \subset Z$ of a topological space is *locally closed* in Z if $Y = F \cap U$ where $F \subset Z$ is closed and $U \subset Z$ is open.

PROPOSITION 2.1. *Let G be a locally compact group, X a Hausdorff topological space and $G \curvearrowright X$ a continuous action. Assume that for every $x \in X$, the orbit Gx is locally closed in X. Then the quotient space $G\backslash X$ is T_0.*

PROOF. Denote by $p : X \to G\backslash X$ the quotient map that is continuous and open. Let $x_1, x_2 \in X$. Assume that $p(x_1)$ and $p(x_2)$ are not separated by an open set of $G\backslash X$. Let $U \subset X$ be an open set such that $x_1 \in U$. Then $p(U) \subset G\backslash X$ is an open set such that $p(x_1) \in p(U)$. It follows that $p(x_2) \in p(U)$ and so $x_2 \in p^{-1}(p(U)) = \bigcup_{g \in G} gU$. This further implies that $x_1 \in \overline{Gx_2}$ and so $Gx_1 \subset \overline{Gx_2}$. Likewise, we have $Gx_2 \subset \overline{Gx_1}$. Then $\overline{Gx_1} = \overline{Gx_2}$. Since $Gx_2 \subset X$ is locally closed, Gx_2 is open in $\overline{Gx_2}$. Since Gx_1 is dense in $\overline{Gx_2}$, it follows that $Gx_1 \cap Gx_2 \neq \emptyset$ and so $Gx_1 = Gx_2$. This shows that $X \backslash G$ is T_0. □

Secondly, we investigate when the quotient space $G\backslash X$ is Hausdorff. We need to introduce some further terminology. We say that a Hausdorff topological space Y is a *$\mathscr{K}$-space* if for any subset $C \subset Y$, we have that C is closed when $C \cap K$ is closed for every compact subset $K \subset Y$. Examples of Hausdorff $\mathscr{K}$-spaces include locally compact spaces and metrizable spaces. A continuous map $f : X \to Y$ is *proper* if $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset Y$.

Lemma 2.2. *Let X be a Hausdorff topological space and Y a Hausdorff topological $\mathscr{K}$-space. Then any continuous proper map $f : X \to Y$ is closed.*

Proof. Let $f : X \to Y$ be a continuous proper map. Let $C \subset X$ be a closed subset. Since Y is a $\mathscr{K}$-space, in order to show that $f(C) \subset Y$ is closed, it suffices to show that $f(C) \cap K$ is closed for every compact subset $K \subset Y$. Let $K \subset Y$ be a compact subset. Since f is proper, $f^{-1}(K)$ is compact and so is $f^{-1}(K) \cap C$. Since f is continuous, $f(f^{-1}(K) \cap C) = f(C) \cap K$ is compact hence closed since Y is Hausdorff. □

We say that a continuous action $G \curvearrowright X$ is *proper* if the continuous map $G \times X : X \times X : (g, x) \mapsto (x, gx)$ is proper. The next proposition provides a sufficient condition for the quotient space $G\backslash X$ to be Hausdorff.

Proposition 2.3. *Let G be a locally compact group, X a Hausdorff topological space such that $X \times X$ is $\mathscr{K}$-space and $G \curvearrowright X$ a proper continuous action. Then the quotient space $G\backslash X$ is Hausdorff. Moreover, for every $x \in X$, the orbit $Gx \subset X$ is closed and the stabilizer subgroup* $\mathrm{Stab}_G(x) < G$ *is compact.*

Proof. Denote by $p : X \to G\backslash X$ the quotient map that is continuous and open. Write $f : G \times X \to X \times X : (g, x) \mapsto (x, gx)$. Since the map f is proper, for every $x \in X$, $\mathrm{Stab}_G(x) \times \{x\} = f^{-1}(\{x, x\}) \subset X \times X$ is compact and so the stabilizer subgroup $\mathrm{Stab}_G(x) < G$ is compact. Since the map f is closed by Lemma 2.2, for every $x \in X$, $f(G \times \{x\}) = \{x\} \times Gx \subset X \times X$ is closed and so $Gx \subset X$ is closed.

Since the map f is closed by Lemma 2.2, $f(G \times X) \subset X \times X$ is closed and so its complement $X \times X \setminus f(G \times X) \subset X \times X$ is open. Since the map $p \times p : X \times X \to G\backslash X \times G\backslash X$ is open, $(p \times p)(X \times X \setminus f(G \times X)) \subset G\backslash X \times G\backslash X$ is open. This further implies that the diagonal

$$\Delta = \{(z, z) \mid z \in G\backslash X\} = G\backslash X \times G\backslash X \setminus (p \times p)(X \times X \setminus f(G \times X))$$

is closed. Thus, the quotient space $G\backslash X$ is Hausdorff. □

In order to state the main result of this subsection, we introduce the following terminology.

DEFINITION 2.4. Let G be a locally compact group. We say that G satisfies the *dynamical dichotomy for isometric actions* if whenever $G \curvearrowright (X, d)$ is a continuous isometric action on a separable metric space, then either there exists $x \in X$ such that $gx = x$ for every $g \in G$ or the action $G \curvearrowright X$ is proper.

The main result of this subsection shows that for every $d \geq 2$, $\mathrm{SL}_d(\mathbb{R})$ satisfies the dynamical dychotomy for isometric actions. More generally, we have the following result.

THEOREM 2.5 (Bader–Gelander [**BG14**]). *Any noncompact simple connected real Lie group G satisfies the dynamical dichotomy for isometric actions.*

We will prove Theorem 2.5 for $G = \mathrm{SL}_d(\mathbb{R})$, $d \geq 2$. First, we need to prove some preliminary results that are also of independent interest. The next easy result is commonly known as Mautner's phenomenon. We refer the reader to [**BG14**] for some historical background.

LEMMA 2.6 (Mautner's phenomenon). *Let G be a locally compact group, (X, d) a metric space and $G \curvearrowright X$ a continuous isometric action. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements in G and $h \in G$ such that $\lim_n g_n h g_n^{-1} = e$. Let $x \in X$ be a point for which $\lim_n g_n x = x$. Then $hx = x$.*

PROOF. We have

$$d(hx, x) = \lim_n d(hg_n^{-1}x, g_n^{-1}x) = \lim_n d(g_n h g_n^{-1}x, x) = d(\lim_n g_n h g_n^{-1}x, x) = 0.$$

Thus, $hx = x$. □

Let $d \geq 2$. For all $1 \leq a \neq b \leq d$ and all $t \in \mathbb{R}$, denote by $E_{ab}(t) \in \mathrm{SL}_d(\mathbb{R})$ the elementary matrix defined by $(E_{ab}(t))_{ij} = 1$ if $i = j$, $(E_{ab}(t))_{ij} = t$ if $i = a$ and $j = b$, $(E_{ab}(t))_{ij} = 0$ otherwise. We leave as an exercise to check that $\mathrm{SL}_d(\mathbb{R})$ is generated by $\{E_{ab}(t) \mid 1 \leq a \neq b \leq d, t \in \mathbb{R}\}$. For every $2 \leq k \leq d$, regard $\mathrm{SL}_k(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$ as the following subgroup:

$$\mathrm{SL}_k(\mathbb{R}) \cong \left\{ \begin{pmatrix} A & 0_{d-k,k} \\ 0_{k,d-k} & 1_{d-k,d-k} \end{pmatrix} \mid A \in \mathrm{SL}_k(\mathbb{R}) \right\} < \mathrm{SL}_d(\mathbb{R}).$$

For all $1 \leq \ell_1 < \ell_2 \leq d$, denote by $H_{\ell_1,\ell_2} < \mathrm{SL}_d(\mathbb{R})$ the (ℓ_1, ℓ_2)-*copy* of $\mathrm{SL}_2(\mathbb{R})$ in $\mathrm{SL}_d(\mathbb{R})$ that consists of all matrices $g \in \mathrm{SL}_d(\mathbb{R})$ such that $g_{\ell_1\ell_1} = \alpha$, $g_{\ell_1\ell_2} = \beta$, $g_{\ell_2\ell_1} = \gamma$, $g_{\ell_2\ell_2} = \delta$, $g_{ii} = 1$ for all $i \neq \ell_1, \ell_2$, $g_{ij} = 0$ for all $i \neq j$ and $\{i, j\} \neq \{\ell_1, \ell_2\}$ and such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

LEMMA 2.7. *Let $d \geq 2$. Let (X, d_X) be a metric space and $\mathrm{SL}_d(\mathbb{R}) \curvearrowright (X, d_X)$ a continuous isometric action. Let $x \in X$ be a H_{ℓ_1,ℓ_2}-fixed point for some $1 \leq \ell_1 < \ell_2 \leq d$. Then $x \in X$ is a global fixed point.*

Proof. Upon permuting the indices, we may assume that $\ell_1 = 1$ and $\ell_2 = 2$. We proceed by induction over $2 \leq k \leq d$. By assumption, $x \in X$ is a $\mathrm{SL}_2(\mathbb{R})$-fixed point. Assume that x is a $\mathrm{SL}_k(\mathbb{R})$-fixed point for $2 \leq k \leq d-1$ and let us show that x is a $\mathrm{SL}_{k+1}(\mathbb{R})$-fixed point. Let $1 \leq j \leq k$ and $t \in \mathbb{R}$. For every $n \geq 1$, denote by $g_n \in \mathrm{SL}_k(\mathbb{R}) < \mathrm{SL}_{k+1}(\mathbb{R})$ any diagonal matrix such that $(g_n)_{ii} = \frac{1}{n}$ if $i = j$. Then a simple computation shows that $g_n E_{j(k+1)}(t) g_n^{-1} = E_{j(k+1)}(\frac{t}{n}) \to 1$ as $n \to \infty$ and $g_n^{-1} E_{(k+1)j}(t) g_n = E_{(k+1)j}(\frac{t}{n}) \to 1$ as $n \to \infty$. Since $g_n x = x$ for every $n \geq 1$, Lemma 2.6 implies that $E_{j(k+1)}(t)x = x$ for every $t \in \mathbb{R}$. Likewise, we have $E_{(k+1)j}(t)x = x$ for every $t \in \mathbb{R}$. Since $\mathrm{SL}_{k+1}(\mathbb{R})$ is generated by $\mathrm{SL}_k(\mathbb{R}) \cup \{E_{j(k+1)}(t), E_{(k+1)j}(t) \mid 1 \leq j \leq k, t \in \mathbb{R}\}$, it follows that x is a $\mathrm{SL}_{k+1}(\mathbb{R})$-fixed point. By induction over $2 \leq k \leq d$, it follows that x is a $\mathrm{SL}_d(\mathbb{R})$-fixed point. $\square$

Let $d \geq 2$. Denote by $K = \mathrm{SO}_d(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$ the special orthogonal subgroup and observe that $K < \mathrm{SL}_d(\mathbb{R})$ is compact. Define the subset $A^+ \subset \mathrm{SL}_d(\mathbb{R})$ of diagonal matrices by

$$A^+ = \{\mathrm{diag}(\lambda_1, \dots, \lambda_d) \mid \lambda_1 \geq \cdots \geq \lambda_d > 0,\ \lambda_1 \cdots \lambda_d = 1\} \subset \mathrm{SL}_d(\mathbb{R})$$

and by $A < \mathrm{SL}_d(\mathbb{R})$ the subgroup of diagonal matrices generated by A^+.

Lemma 2.8 (Cartan decomposition). *We have* $\mathrm{SL}_d(\mathbb{R}) = K \cdot A^+ \cdot K$.

Proof. Let $g \in \mathrm{SL}_d(\mathbb{R})$. By polar decomposition, we may write $g = k_0 h$ where $k_0 \in K$ and $h \in \mathrm{SL}_d(\mathbb{R})$ is symmetric positive definite. By diagonalization, there exists $k_2 \in K$ such that $k_2 h k_2^{-1} = a \in A^+$. Then $g = k_1 a k_2$ with $k_1 = k_0 k_2^{-1} \in K$. $\square$

We now have all the tools to prove Theorem 2.5.

Proof of Theorem 2.5. Let $d \geq 2$ and write $G = \mathrm{SL}_d(\mathbb{R})$. Let (X, d_X) be a separable metric space and $G \curvearrowright (X, d_X)$ a continuous isometric action. Assuming that the action $G \curvearrowright X$ is not proper, we show that there exists a global fixed point. Since G is second countable and X is a separable metric space and since the map $G \times X \to X \times X : (g, x) \mapsto (x, gx)$ is not proper, there exist a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $g_n \to \infty$ in G, a sequence $(x_n)_{n\in\mathbb{N}}$ in X and $x, y \in X$ such that $\lim_n (x_n, g_n x_n) = (x, y)$ in $X \times X$. Using Lemma 2.8, there exist sequences $(k_{1,n})_{n\in\mathbb{N}}$ and $(k_{2,n})_{n\in\mathbb{N}}$ in K and $(a_n)_{n\in\mathbb{N}}$ in A^+ such that $g_n = k_{1,n} a_n k_{2,n}$ for every $n \in \mathbb{N}$. Upon taking a subsequence, we may further assume that $k_{1,n} \to k_1$ in K and $k_{2,n} \to k_2$ in K. Set $x_1 = k_1^{-1} y \in X$ and $x_2 = k_2 x \in X$. Choose an increasing function $\psi : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\psi(n)} a_n^{-1} \to \infty$ in G. Since $G \curvearrowright (X, d)$ is continuous and isometric, we have $\lim_n a_n x_2 = x_1$, $\lim_n a_n^{-1} x_1 = x_2$ and $\lim_n b_n x_1 = x_1$.

For every $n \in \mathbb{N}$, upon conjugating b_n by an element in K and using again the fact that K is compact, we may assume that $b_n = \mathrm{diag}(\lambda_{1,n}, \dots, \lambda_{d,n}) \in$

A^+ with $\lambda_{1,n} \geq \cdots \geq \lambda_{d,n}$ and $\lambda_{1,n} \cdots \lambda_{d,n} = 1$. Since $b_n \to \infty$, it follows that $\frac{\lambda_{1,n}}{\lambda_{d,n}} \to +\infty$. A simple computation shows that for every $t \in \mathbb{R}$,

$$b_n^{-1} E_{1d}(t) b_n = E_{1d}(\frac{\lambda_{d,n}}{\lambda_{1,n}} t) \to 1 \quad \text{as} \quad n \to \infty.$$

Then Lemma 2.6 implies that $E_{1d}(t)x_1 = x_1$ for every $t \in \mathbb{R}$. Likewise, we have $E_{d1}(t)x_1 = x_1$ for every $t \in \mathbb{R}$. Then Lemma 2.7 further implies that $x_1 \in X$ is a global fixed point. This finishes the proof of Theorem 2.5. $\square$

1.2. Measurable dynamics. In this subsection, we assume that the group G is locally compact second countable. We endow G with its σ-algebra $\mathscr{B}(G)$ of Borel subsets. We fix a left invariant Haar measure m_G on G. Let X be a standard Borel space and denote by $\mathrm{Prob}(X)$ the standard Borel space of all Borel probability measures on X. We say that the action $G \curvearrowright X$ is *Borel* if the action map $G \times X \to X : (g, x) \mapsto gx$ is Borel. We say that the action $G \curvearrowright X$ is *tame* if the quotient Borel space $G \backslash X$ is *countably separated*. Recall that a Borel space Z is *countably separated* if there exists a countable family $(U_n)_{n \in \mathbb{N}}$ of Borel subsets of Z that separates the points in Z in the following sense: for every $z_1, z_2 \in Z$ such that $z_1 \neq z_2$, there exists $n \in \mathbb{N}$ such that $z_1 \in U_n$ and $z_2 \notin U_n$ or $z_1 \notin U_n$ and $z_2 \in U_n$. If the Borel action $G \curvearrowright X$ is tame, then the quotient Borel space $G \backslash X$ is standard by Theorem A.1. We record the following consequence of Proposition 2.1.

PROPOSITION 2.9. *Let G be a locally compact second countable group, X a Polish space and $G \curvearrowright X$ a continuous action. Assume that for every $x \in X$, the orbit Gx is locally closed in X. Then the Borel action $G \curvearrowright X$ is tame and the quotient Borel space $G \backslash X$ is standard.*

PROOF. By Proposition 2.1, the quotient space $G \backslash X$ is T_0. Since X is a Polish space, there is a countable basis of open sets that generates the topology on $G \backslash X$. Therefore, the Borel space $G \backslash X$ is countably separated and so the Borel action $G \curvearrowright X$ is tame and the quotient Borel space $G \backslash X$ is standard by Theorem A.1. $\square$

Let $\nu \in \mathrm{Prob}(X)$ and assume that for every $g \in G$, the measures ν and $g_*\nu$ are equivalent on X. In that case, we say that the action $G \curvearrowright (X, \nu)$ is *nonsingular*. Recall that $\mathrm{L}^\infty(X, \nu) = \mathrm{L}^1(X, \nu)^*$ so that $\mathrm{L}^\infty(X, \nu)$ is also endowed with the weak*-topology. By [**Ru91**, Theorem 3.10], we may identify $\mathrm{L}^1(X, \nu)$ with the space of all weak*-continuous linear functionals on $\mathrm{L}^\infty(X, \nu)$. Any nonsingular action $G \curvearrowright (X, \nu)$ gives rise to an action $\alpha : G \curvearrowright \mathrm{L}^\infty(X, \nu)$ defined by the formula

$$\forall g \in G, \forall F \in \mathrm{L}^\infty(X, \nu), \quad \alpha(g)(F) = F \circ g^{-1}.$$

The action map $G \times \mathrm{L}^\infty(X, \nu) \to \mathrm{L}^\infty(X, \nu) : (g, F) \mapsto \alpha(g)(F)$ is separately continuous when $\mathrm{L}^\infty(X, \nu)$ is endowed with the weak*-topology. This follows from the fact that the action $G \curvearrowright \mathrm{L}^1(X, \nu)$ is $\| \cdot \|_1$-continuous. We will then simply say that the action $\alpha : G \curvearrowright \mathrm{L}^\infty(X, \nu)$ is weak*-continuous. We refer the reader to [Ta03, Proposition XIII.1.2] for further details. For every Borel probability measure $\eta \in \mathrm{Prob}(X)$ such that $\eta \prec \nu$, me may regard $\eta \in \mathrm{L}^1(X,\nu)$ and we simply denote by $\eta : \mathrm{L}^\infty(X,\nu) \to \mathbb{C} : f \mapsto \int_X f(x)\,\mathrm{d}\eta(x)$ the corresponding weak*-continuous positive unital linear functional. When the context is clear, we will often simply write $\mathrm{L}^\infty(X) = \mathrm{L}^\infty(X,\nu)$.

In these notes, we will be particularly interested in nonsingular actions arising from homogeneous spaces.

Theorem 2.10. *Let G be a locally compact second countable group and $H < G$ a closed subgroup. Then the quotient space $G/H = \{gH \mid g \in G\}$ endowed with the quotient topology is Hausdorff locally compact second countable. The left translation action $G \curvearrowright G/H$ is continuous and transitive. Moreover, G/H carries a unique G-invariant measure class.*

Proof. Simply denote by $p : G \to G/H : g \mapsto gH$ the quotient map. Firstly, we show that the right multiplication action $H \curvearrowright G$ is proper. Indeed, consider the continuous map $f : H \times G \to G \times G : (h,g) \mapsto (g, gh^{-1})$. For every compact subset $K \subset G \times G$, we may choose a compact subset $L \subset G$ such that $K \subset L \times L$. Then we have

$$f^{-1}(K) \subset f^{-1}(L \times L) \subset L^{-1}L \times L$$

and so $f^{-1}(K) \subset H \times G$ is compact. Then Proposition 2.3 implies that the quotient space $G/H = \{gH \mid g \in G\}$ is Hausdorff. Since the quotient map $p : G \to G/H$ is continuous and open and since G is locally compact second countable, it follows that G/H is locally compact second countable.

Next, define the action map $a : G \times G/H \to G/H : (g,c) \mapsto gc$. Recall that the multiplication map $m : G \times G \to G$ is continuous. Since the map $\mathrm{id}_G \times p : G \times G \to G \times G/H : (g,h) \mapsto (g, hH)$ is continuous and open, the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\ m\ } & G \\ \Big\downarrow{\scriptstyle \mathrm{id}_G \times p} & & \Big\downarrow{\scriptstyle p} \\ G \times G/H & \xrightarrow{\ a\ } & G/H \end{array}$$

shows that the action map $a : G \times G/H \to G/H$ is continuous.

Finally, we show that G/H carries a unique G-invariant measure class. Fix a Borel probability measure $\mu \in \mathrm{Prob}(G)$ that is equivalent to the left Haar measure m_G and set $\nu = p_*\mu \in \mathrm{Prob}(G/H)$. For every $g \in G$, since $g_*\mu \sim \mu$, we have $g_*\nu \sim \nu$. This implies that the measure class of ν is G-invariant. Let now $\eta \in \mathrm{Prob}(G/H)$ be a Borel probability measure such that $g_*\eta \sim \eta$ for every $g \in G$. We prove the following claim.

Claim 2.11. For every Borel function $f : G/H \to \mathbb{R}_+$, the following assertions are equivalent:

(i) $\eta(f) = 0$.
(ii) For every $c \in G/H$, we have $\int_G f(gc)\,\mathrm{d}m_G(g) = 0$.

Proof of Claim 2.11. (i) ⇒ (ii) For every $g \in G$, since $g_*\eta \sim \eta$ and since $\eta(f) = 0$, we have

$$\int_{G/H} f(gc)\,\mathrm{d}\eta(c) = \int_{G/H} f(c)\frac{\mathrm{d}g_*\eta}{\mathrm{d}\eta}(c)\,\mathrm{d}\eta(c) = 0$$

Applying Fubini's theorem, we have

$$\int_{G/H}\int_G f(gc)\,\mathrm{d}m_G(g)\,\mathrm{d}\eta(c) = \int_G\int_{G/H} f(gc)\,\mathrm{d}\eta(c)\,\mathrm{d}m_G(g) = 0.$$

This implies that there exists $c \in G/H$ such that $\int_G f(gc)\,\mathrm{d}m_G(g) = 0$. Then for every $h \in G$, we have

$$\int_G f(ghc)\,\mathrm{d}m_G(g) = \Delta_G(h^{-1})\int_G f(gc)\,\mathrm{d}m_G(g) = 0.$$

Since $G \curvearrowright G/H$ is transitive, this shows that for every $c \in G/H$, we have $\int_G f(gc)\,\mathrm{d}m_G(g) = 0$.

(ii) ⇒ (i) Applying Fubini's theorem, we have

$$\int_G\int_{G/H} f(gc)\,\mathrm{d}\eta(c)\,\mathrm{d}m_G(g) = \int_{G/H}\int_G f(gc)\,\mathrm{d}m_G(g)\,\mathrm{d}\eta(c) = 0.$$

This implies that there exists $g \in G$ such that $\int_{G/H} f(gc)\,\mathrm{d}\eta(c) = 0$. Since $g_*^{-1}\eta \sim \eta$, it follows that

$$\eta(f) = \int_{G/H} f(c)\,\mathrm{d}\eta(c) = \int_{G/H} f(gc)\frac{\mathrm{d}g_*^{-1}\eta}{\mathrm{d}\eta}(c)\,\mathrm{d}\eta(c) = 0.$$

This finishes the proof of the claim. □

Observe that item (ii) in Claim 2.11 does not depend on the choice of the G-quasi-invariant measure $\eta \in \mathrm{Prob}(G/H)$. Therefore, for every Borel function $f : G/H \to \mathbb{R}_+$, we have $\eta(f) = 0$ if and only if $\nu(f) = 0$. This shows that $\eta \sim \nu$. Thus, there is a unique G-invariant measure class on G/H. □

Next, exploiting the structure of homogeneous space, we record the following useful fact due to Effros (see e.g. [**Zi84**, Lemma 2.1.15]).

Proposition 2.12. *Let G be a locally compact second countable group, X a Polish space and $G \curvearrowright X$ a continuous action. For every $x \in X$, the following assertions are equivalent:*

(i) *The orbit Gx is locally closed in X.*
(ii) *The continuous map $G/\mathrm{Stab}_G(x) \to Gx : g\,\mathrm{Stab}_G(x) \mapsto gx$ is a homeomorphism when $G/\mathrm{Stab}_G(x)$ is endowed with the quotient topology and $Gx \subset X$ is endowed with the relative topology.*

Proof. Set $Y = \overline{Gx} \subset X$. Then Y is a Polish space, $G \curvearrowright Y$ is continuous and Gx is dense in Y. Since $\mathrm{Stab}_G(x) < G$ is a closed subgroup, Theorem 2.10 implies that the homogeneous space $G/\mathrm{Stab}_G(x)$ endowed with the quotient topology is Hausdorff locally compact second countable.

(i) $\Rightarrow$ (ii) The map $\zeta : G/\operatorname{Stab}_G(x) \to Gx : g\operatorname{Stab}_G(x) \mapsto gx$ is continuous and bijective. In order to prove that $\zeta : G/\operatorname{Stab}_G(x) \to Gx$ is a homeomorphism, it suffices to prove that $\zeta : G/\operatorname{Stab}_G(x) \to Gx$ is open. Thus, it suffices to prove that the orbit map $G \to Gx : g \mapsto gx$ is open. Let $V \subset G$ be an open neighborhood of $e \in G$. Choose a symmetric compact neighborhood $U \subset G$ of $e \in G$ such that $U^2 \subset V$. We claim that Ux has nonempty interior. Indeed, choose a countable dense subset $\{g_n \mid n \in \mathbb{N}\}$ of G and observe that $\bigcup_{n\in\mathbb{N}} g_n U = G$. Then $\bigcup_{n\in\mathbb{N}} g_n Ux = Gx$. Since $Gx < Y$ is open and since for every $n \in \mathbb{N}$, $g_n Ux$ is compact hence closed, Baire's property implies that there exists $n \in \mathbb{N}$ such that $g_n Ux$ has nonempty interior. Thus, Ux has nonempty interior. Choose $u \in U$ such that Ux is a neighborhood of ux. Then $u^{-1}Ux$ is a neighborhood of $x \in X$ and so Vx is a neighborhood of $x \in X$. This shows that the orbit map $G \to Gx$ is open and so $\zeta : G/\operatorname{Stab}_G(x) \to Gx$ is a homeomorphism.

(ii) $\Rightarrow$ (i) Since $G/\operatorname{Stab}_G(x)$ is a Hausdorff locally compact second countable space and since the map $G/\operatorname{Stab}_G(x) \to Gx$ is a homeomorphism, it follows that Gx satisfies Baire's property with respect to the relative topology. Since G is σ-compact, there exists a sequence of compact subsets $K_n \subset G$ such that $G = \bigcup_{n\in\mathbb{N}} K_n$. Then $Gx = \bigcup_{n\in\mathbb{N}} K_n x$ and for every $n \in \mathbb{N}$, $K_n x \subset Gx$ is compact hence closed. Then there exists $n \in \mathbb{N}$ such that $K_n x$ has a nonempty interior with respect to the relative topology. In particular, there exists an open set $V \subset Y$ such that $V \cap Gx \subset K_n x$. Since $Gx \subset Y$ is dense and since $V \subset Y$ is open, we have

$$V = V \cap \overline{Gx} \subset \overline{V \cap Gx} \subset K_n x.$$

This further implies that $Gx = GV$ and so $Gx \subset Y$ is open. □

More generally, we state the following characterization due to Effros (see [**Zi84**, Theorem 2.14] for a proof).

THEOREM 2.13. *Let G be a locally compact second countable group, X a Polish space and $G \curvearrowright X$ a continuous action. The following assertions are equivalent:*

(i) *For every $x \in X$, the orbit Gx is locally closed in X.*
(ii) *For every $x \in X$, the map $G/\operatorname{Stab}_G(x) \to Gx : g\operatorname{Stab}_G(x) \mapsto gx$ is a homeomorphism.*
(iii) *The Borel action $G \curvearrowright X$ is tame.*

Let now $G \curvearrowright (X, \nu)$ be a nonsingular action. Let $Y \subset X$ be a measurable subset. We say that $Y \subset X$ is *G-invariant* if for every $g \in G$, we have $\nu(gY \triangle Y) = 0$. Let $f : X \to \mathbb{C}$ be a measurable map. We say that $f : X \to \mathbb{C}$ is *G-invariant* if for every $g \in G$ and ν-almost every $x \in X$, we have $f(gx) = f(x)$. The next lemma shows that a G-invariant measurable subset (resp. function) coincides ν-almost everywhere with a strictly G-invariant measurable subset (resp. function).

LEMMA 2.14. *Let $G \curvearrowright (X, \nu)$ be a nonsingular action. The following assertions hold:*

(i) *For any G-invariant measurable subset $Y \subset X$, there is a strictly G-invariant measurable subset $Z \subset X$ such that $\nu(Y \triangle Z) = 0$.*
(ii) *For any G-invariant measurable function $f : X \to \mathbb{C}$, there is a strictly G-invariant measurable function $F : X \to \mathbb{C}$ such that $\nu(\{f \neq F\}) = 0$.*

PROOF. Since the proofs of items (i) and (ii) are analogous, we only prove item (i). Fix a left invariant Haar measure m_G on G. By assumption and using Fubini's theorem, the measurable subset

$$X_0 = \left\{x \in X \mid G \to [0,1] : g \mapsto \mathbf{1}_Y(g^{-1}x) \text{ is } m_G\text{-a.e. constant}\right\}$$

is conull in X. For every $x \in X_0$, denote by $f(x)$ the unique essential value of the measurable function $G \to [0,1] : g \mapsto \mathbf{1}_Y(g^{-1}x)$. For every $x \in X \setminus X_0$, set $f(x) = 0$. Note that $f(X) \subset \{0,1\}$. Fubini's theorem implies that the function $f : X \to [0,1]$ is measurable and $f(x) = \mathbf{1}_Y(x)$ for ν-almost every $x \in X$. For every $x \in X_0$ and every $h \in G$, the measurable function $G \to [0,1] : g \mapsto \mathbf{1}_Y(g^{-1}h^{-1}x)$ is m_G-almost everywhere constant, hence $h^{-1}x \in X_0$ and $f(h^{-1}x) = f(x)$. This further implies that $X_0 \subset X$ is strictly G-invariant and f is strictly G-invariant in the sense that $f(g^{-1}x) = f(x)$ for every $g \in G$ and every $x \in X$. Set $Z = \{x \in X \mid f(x) = 1\}$. Then $Z \subset X$ is a strictly G-invariant measurable subset such that $\nu(Y \triangle Z) = 0$. $\square$

PROPOSITION 2.15. *Let $G \curvearrowright (X, \nu)$ be a nonsingular action. The following assertions are equivalent:*

(i) *Every G-invariant measurable subset $Y \subset X$ is null or conull.*
(ii) *Every G-invariant measurable function $f : X \to \mathbb{C}$ is ν-almost everywhere constant.*

PROOF. (i) $\Rightarrow$ (ii) By contraposition, assume that there exists a G-invariant measurable function $f : X \to \mathbb{C}$ that is not ν-almost everywhere constant. Upon taking the real or imaginary part of f, we may assume that $f(X) \subset \mathbb{R}$. Next, upon taking $f^+ = \max(f, 0)$ or $f^- = \max(-f, 0)$, we may further assume that $f(X) \subset \mathbb{R}_+$. For every $t > 0$, define the G-invariant measurable subset $X_t = \{x \in X \mid f(x) \geq t\}$. By Fubini's theorem, the function $\mathbb{R}_+^* \to \mathbb{R}_+ : t \mapsto \nu(X_t)$ is measurable, nonincreasing and satisfies $\int_X f(x) \, \mathrm{d}\nu(x) = \int_0^{+\infty} \nu(X_t) \, \mathrm{d}t$. We claim that there exists $t > 0$ such that $0 < \nu(X_t) < 1$. Indeed, otherwise there would exist $s > 0$ such that $\nu(X_t) = 0$ for every $t > s$ and $\nu(X_t) = 1$ for every $t \leq s$. This would imply that $f = s$ ν-almost everywhere, a contradiction. Therefore, there exists $t > 0$ such that $0 < \nu(X_t) < 1$. This shows the existence of a G-invariant measurable subset $Y = X_t \subset X$ that is neither null nor conull.

(ii) $\Rightarrow$ (i) Let $Y \subset X$ be a G-invariant measurable subset. Then the measurable function $f = \mathbf{1}_Y$ is G-invariant whence ν-almost everywhere constant. If $f = 0$ ν-almost everywhere, then $Y \subset X$ is null. If $f = 1$ ν-almost everywhere, then $Y \subset X$ is conull. $\square$

Definition 2.16. We say that the nonsingular action $G \curvearrowright (X, \nu)$ is

- *ergodic* if $G \curvearrowright (X, \nu)$ satisfies one of the equivalent conditions in Proposition 2.15.
- *doubly ergodic* if the diagonal action $G \curvearrowright (X \times X, \nu \otimes \nu)$ is ergodic.

Let Z be a standard Borel space and $G \curvearrowright Z$ a Borel action. Let $f : X \to Z$ be a measurable map. We say that f is *G-equivariant* if for every $g \in G$ and ν-almost every $x \in X$, we have $f(gx) = gf(x)$. The next lemma shows that any G-equivariant measurable map coincides ν-almost everywhere with a strictly G-equivariant measurable map.

Lemma 2.17. *For any G-equivariant measurable map $f : X \to Z$, there is a conull strictly G-invariant measurable subset $X_0 \subset X$ and a strictly G-equivariant measurable map $F : X_0 \to Z$ such that $f = F$ ν-almost everywhere.*

Proof. Fix a left invariant Haar measure m_G on G. We may regard $Z \subset [0, 1]$ as a Borel subset. By assumption and using Fubini's theorem, the measurable subset

$$X_0 = \left\{x \in X \mid G \to Z : g \mapsto g^{-1} f(gx) \text{ is } m_G\text{-a.e. constant}\right\}$$

is conull in X. For every $x \in X_0$, denote by $F(x)$ the unique essential value of the measurable function $G \to Z : g \mapsto g^{-1} f(gx)$. Fubini's theorem implies that the function $F : X_0 \to Z$ is measurable and $f = F$ ν-almost everywhere. For every $x \in X_0$ and every $h \in G$, the measurable function $G \to Z : g \mapsto (gh)^{-1} f(ghx)$ is m_G-almost everywhere constant, hence $hx \in X_0$ and $h^{-1}F(hx) = F(x)$. This further implies that $X_0 \subset X$ is strictly G-invariant and $F : X_0 \to Z$ is strictly G-equivariant in the sense that $F(gx) = gF(x)$ for every $g \in G$ and every $x \in X_0$. □

The following terminology due to Bader–Furman [**BF14**] will be crucial in these notes.

Definition 2.18 (Metric ergodicity). We say that the nonsingular action $G \curvearrowright (X, \nu)$ is *metrically ergodic* if for every separable metric space (Z, d) and every continuous isometric action $G \curvearrowright (Z, d)$, every G-equivariant measurable map $f : X \to Z$ is ν-almost everywhere constant.

One of the main examples of metrically ergodic actions arise from homogeneous spaces associated with locally compact groups satisfying the dynamical dichotomy for isometric ations.

Proposition 2.19. *Let G be a locally compact second countable group satisfying the dynamical dichotomy for isometric ations. Let $H < G$ be a noncompact closed subgroup. Then the action $G \curvearrowright G/H$ is metrically ergodic.*

Proof. Let $G \curvearrowright (Z, d)$ be a continuous isometric action on a separable metric space. Let $f : G/H \to Z$ be a G-equivariant measurable map.

By Lemma 2.17 and since $G \curvearrowright G/H$ is transitive, we may assume that $f : G/H \to Z$ is strictly G-equivariant. Set $z = f(H) \in Z$. For every $g \in G$, we have $f(gH) = gf(H) = gz$ so that $f(G/H) = Gz$. Moreover, for every $h \in H$, we have $hz = f(hH) = f(H) = z$. Applying the dynamical dichotomy to the continuous isometric action $G \curvearrowright (Gz, d)$ and since $H < G$ is noncompact and $H < \mathrm{Stab}_G(z)$, it follows that $\mathrm{Stab}_G(z) = G$. Then for every $g \in G$, we have $f(gH) = gf(H) = gz = z$ and so $f : G/H \to Z$ is constant. This shows that $G \curvearrowright G/H$ is metrically ergodic. $\square$

Next, we show that metrically ergodic actions are stable under taking restrictions to lattices.

Proposition 2.20. *Let G be a locally compact second countable group and $\Gamma < G$ a lattice. Then for any metrically ergodic action $G \curvearrowright (X, \nu)$, the restriction $\Gamma \curvearrowright (X, \nu)$ is metrically ergodic.*

Proof. Let (Z, d_Z) be a separable metric space and $\Gamma \curvearrowright (Z, d_Z)$ an isometric action. Let $f : X \to Z$ be a Γ-equivariant measurable map. We need to show that f is ν-almost everywhere constant.

In order to do so, we define the *induced metric space* $(\mathscr{Z}, d_{\mathscr{Z}})$ as follows. As usual, denote by m_G a Haar measure on G. Denote by $m_{G/\Gamma}$ the unique G-invariant Borel probability measure on G/Γ. Upon replacing d_Z by $\min(d_Z, 1)$, we may assume that d_Z is bounded on Z. Define $\mathscr{Z}$ to be the space of all m_G-equivalence classes of measurable maps $F : G \to Z$ that are right Γ-equivariant in the sense that for m_G-almost every $g \in G$ and every $\gamma \in \Gamma$, we have $F(g\gamma^{-1}) = \gamma F(g)$. Observe that for all $F_1, F_2 \in \mathscr{Z}$, the measurable function $G \to \mathbb{R}_+ : g \mapsto d_Z(F_1(g), F_2(g))$ is right Γ-invariant. We may then endow the space $\mathscr{Z}$ with the metric $d_{\mathscr{Z}}$ defined by

$$\forall F_1, F_2 \in \mathscr{Z}, \quad d_{\mathscr{Z}}(F_1, F_2) = \int_{G/\Gamma} d_Z(F_1(g), F_2(g))\, \mathrm{d}m_{G/\Gamma}(g\Gamma).$$

Then $(\mathscr{Z}, d_{\mathscr{Z}})$ is a separable metric space. Define the action $G \curvearrowright \mathscr{Z}$ by $gF : G \to Z : h \mapsto F(g^{-1}h)$ for every $g \in G$ and every $F \in \mathscr{Z}$. We prove the following claim.

Claim 2.21. The action $G \curvearrowright (\mathscr{Z}, d_{\mathscr{Z}})$ is continuous and isometric.

Proof of Claim 2.21. It is plain to see that $G \curvearrowright (\mathscr{Z}, d_{\mathscr{Z}})$ is isometric. It remains to prove that $G \curvearrowright (\mathscr{Z}, d_{\mathscr{Z}})$ is continuous. It suffices to prove that for every $F \in \mathscr{Z}$, the map $G \to \mathbb{R}_+ : g \mapsto d_{\mathscr{Z}}(gF, F)$ is continuous at $e \in G$.

Let $F \in \mathscr{Z}$. Fubini's theorem implies that the map $G \to \mathbb{R}_+ : g \mapsto d_{\mathscr{Z}}(gF, F)$ is measurable. Let $\varepsilon > 0$ and set $B = \{g \in G \mid d_{\mathscr{Z}}(gF, F) < \varepsilon/2\}$. Then $B \subset G$ is a measurable subset such that $B^{-1} = B$ and $B^2 = BB^{-1} \subset \{g \in G \mid d_{\mathscr{Z}}(gF, F) < \varepsilon\}$. Since $\mathscr{Z}$ is separable, there exists a sequence $(g_n)_{n \in \mathbb{N}}$ in G such that $\{g_n F \mid n \in \mathbb{N}\}$ is dense in $\{gF \mid g \in G\}$. This implies that $\bigcup_{n \in \mathbb{N}} g_n B = G$ and so $m_G(B) > 0$. Since G is σ-compact, upon replacing B by $B \cap K$ for a suitable symmetric compact subset, we may further assume that $B = B^{-1}$, $B \subset K$ and $0 < m_G(B) < +\infty$. Then $\mathbf{1}_B \in \mathrm{L}^2(G, \mathscr{B}(G), m_G)$ and $\varphi = \mathbf{1}_B * \mathbf{1}_B \in \mathrm{C}_c(G)$. Since $\varphi(e) = m_G(B) > 0$, the subset $U = \varphi^{-1}(0, +\infty)$ is open, $e \in U$ and $U \subset B^2 \subset \{g \in G \mid d_{\mathscr{Z}}(gF, F) < \varepsilon\}$. This shows that the map $G \to \mathbb{R}_+ : g \mapsto d_{\mathscr{Z}}(gF, F)$ is continuous at $e \in G$. □

We now have all the tools to show that f is ν-almost everywhere constant. Define the G-equivariant measurable map $\widehat{f} : X \to \mathscr{Z} : x \mapsto \big(g \mapsto f(g^{-1}x)\big)$. Since $G \curvearrowright (X, \nu)$ is metrically ergodic, it follows that $\widehat{f}$ is ν-almost everywhere constant. Then there exists $F \in \mathscr{Z}$ such that $\widehat{f}(x) = F$ for ν-almost every $x \in X$. Fubini's theorem implies that there exists $g \in G$ such that $f(g^{-1}x) = F(g)$ for ν-almost every $x \in X$. Thus, $f : X \to Z$ is ν-almost everywhere constant. □

The following proposition clarifies the relations between the various notions of ergodicity we have introduced so far.

Proposition 2.22. *Any doubly ergodic action is metrically ergodic. Any metrically ergodic action is ergodic.*

Proof. It is obvious that any metrically ergodic action is ergodic. Thus, we only prove that any doubly ergodic action is metrically ergodic. Let $G \curvearrowright (X, \nu)$ be a doubly ergodic action. Let $G \curvearrowright (Z, d)$ be a continuous isometric action on a separable metric space. Let $f : X \to Z$ be a G-equivariant measurable map. Define the G-invariant measurable map $X \times X \to \mathbb{R}_+ : (x, y) \mapsto d(f(x), f(y))$. Since $G \curvearrowright (X, \nu)$ is doubly ergodic, there exists $\alpha \geq 0$ such that $d(f(x), f(y)) = \alpha$ for $\nu \otimes \nu$-almost every $(x, y) \in X \times X$. We claim that $\alpha = 0$. Indeed, otherwise assume that $\alpha > 0$. Choose an essential value $z \in Z$ of the measurable $f : X \to Z$. Denote by $B(z, \alpha/2)$ the open ball in Z of center z and radius $\alpha/2$. Define the measurable subset $U = f^{-1}(B(z, \alpha/2)) \subset X$ and observe that $\nu(U) > 0$. By the triangle inequality, for every $(x, y) \in U \times U$, we have $d(f(x), f(y)) \leq d(f(x), z) + d(z, f(y)) < \alpha/2 + \alpha/2 = \alpha$. This is a contradiction. Thus $\alpha = 0$. By Fubini's theorem, we may choose $x \in X$ such that $f(y) = f(x)$ for ν-almost every $y \in X$. Thus, f is ν-almost everywhere constant. This shows that $G \curvearrowright (X, \nu)$ is metrically ergodic. □

In the next section, we will provide a characterization of doubly ergodic probability measure preserving (pmp) actions and we will prove that double ergodicity and metric ergodicity are equivalent for pmp actions.

2. Unitary representations

2.1. Generalities. Let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be a (complex) Hilbert space. We always assume that $\langle \cdot, \cdot \rangle$ is conjugate linear in the second variable. Denote by $\mathrm{B}(\mathscr{H})$ the unital Banach $*$-algebra of all bounded linear operators $T :$

$\mathscr{H} \to \mathscr{H}$. Besides the norm topology on $\mathrm{B}(\mathscr{H})$ given by the supremum norm

$$\forall T \in \mathrm{B}(\mathscr{H}), \quad \|T\|_\infty = \sup\{\|T\xi\| \mid \xi \in \mathscr{H}, \|\xi\| \leq 1\},$$

we can define two weaker locally convex Hausdorff topologies on $\mathrm{B}(\mathscr{H})$ as follows.

- The *strong operator topology* on $\mathrm{B}(\mathscr{H})$ is defined as the initial topology on $\mathrm{B}(\mathscr{H})$ that makes the maps $\mathrm{B}(\mathscr{H}) \to \mathscr{H} : T \mapsto T\xi$ continuous for all $\xi \in \mathscr{H}$.
- The *weak operator topology* on $\mathrm{B}(\mathscr{H})$ is defined as the initial topology on $\mathrm{B}(\mathscr{H})$ that makes the maps $\mathrm{B}(\mathscr{H}) \to \mathbb{C} : T \mapsto \langle T\xi, \eta\rangle$ continuous for all $\xi, \eta \in \mathscr{H}$.

Observe that when $\mathscr{H}$ is separable, both strong and weak operator topologies are metrizable on the unit ball of $\mathrm{B}(\mathscr{H})$ denoted by $\mathrm{Ball}(\mathrm{B}(\mathscr{H}))$. Moreover, $\mathrm{Ball}(\mathrm{B}(\mathscr{H}))$ is weakly compact. We also denote by

$$\mathscr{U}(\mathscr{H}) = \{u \in \mathrm{B}(\mathscr{H}) \mid u^*u = uu^* = 1_{\mathscr{H}}\}$$

the group of unitary operators on $\mathscr{H}$. We simply write $1 = 1_{\mathscr{H}}$. On $\mathscr{U}(\mathscr{H})$, strong and weak operator topologies coincide. Then $\mathscr{U}(\mathscr{H})$ is a topological group but $\mathscr{U}(\mathscr{H})$ need not be locally compact. When $\mathscr{H}$ is separable, $\mathscr{U}(\mathscr{H})$ is a Polish group.

Choose an orthonormal basis $(e_i)_i$ on $\mathscr{H}$ and denote by

$$\mathrm{Tr} : \mathrm{B}(\mathscr{H})_+ \to \mathbb{R}_+ : T \mapsto \sum_{i \in I} \langle Te_i e_i\rangle$$

the associated trace. Denote by

$$\mathrm{HS}(\mathscr{H}) = \{T \in \mathrm{B}(\mathscr{H}) \mid \mathrm{Tr}(T^*T) < +\infty\}$$

the space of Hilbert–Schmidt operators on $\mathscr{H}$. Then the space $\mathrm{HS}(\mathscr{H})$ endowed with the inner product defined by $\langle S, T\rangle_{\mathrm{HS}} = \mathrm{Tr}(T^*S)$ for all $S, T \in \mathrm{HS}(\mathscr{H})$ is a Hilbert space. Denote by $\overline{\mathscr{H}}$ the conjugate Hilbert space and consider the tensor product Hilbert space $\mathscr{H} \otimes \overline{\mathscr{H}}$. Then the mapping

$$\mathscr{H} \otimes_{\mathrm{alg}} \overline{\mathscr{H}} \to \mathrm{HS}(\mathscr{H}) : \xi \otimes \eta \mapsto \langle \,\cdot\,, \eta\rangle\, \xi$$

extends to a well-defined unitary operator $W : \mathscr{H} \otimes \overline{\mathscr{H}} \to \mathrm{HS}(\mathscr{H})$. Moreover, for every $u \in \mathscr{U}(\mathscr{H})$ and every $\zeta \in \mathscr{H} \otimes \overline{\mathscr{H}}$, we have $W(u\zeta) = uW(\zeta)u^*$.

DEFINITION 2.23. Let G be a locally compact group. We say that the mapping $\pi : G \to \mathscr{U}(\mathscr{H}_\pi)$ is a *strongly continuous unitary representation* if the following conditions hold:

(i) $\pi : G \to \mathscr{U}(\mathscr{H}_\pi)$ is a group homomorphism.
(ii) $\pi : G \to \mathscr{U}(\mathscr{H}_\pi)$ is strongly continuous, meaning that π is a continuous map when $\mathscr{U}(\mathscr{H}_\pi)$ is endowed with the strong operator topology as above.

When $\pi : G \to \mathscr{U}(\mathscr{H}_\pi)$ only satisfies condition (i), we simply say that π is a *unitary representation*. When G is discrete, condition (ii) is trivially satisfied.

The next result shows that in order to prove that the unitary representation $\pi : G \to \mathscr{U}(\mathscr{H}_\pi)$ is strongly continuous, it is enough to show that the coefficients of π are measurable functions.

Lemma 2.24. *Let G be a locally compact group, $\mathscr{H}_\pi$ a separable Hilbert space and $\pi : G \to \mathscr{U}(\mathscr{H}_\pi)$ a unitary representation. Assume that for all $\xi, \eta \in \mathscr{H}_\pi$, the map $\varphi_{\xi,\eta} : G \to \mathbb{C} : g \mapsto \langle \pi(g)\xi, \eta\rangle$ is measurable. Then π is strongly continuous.*

Proof. Let $\xi \in \mathscr{H}_\pi$. It suffices to show that the map $G \to \mathscr{H}_\pi : g \mapsto \pi(g)\xi$ is continuous at $e \in G$. Let $Q \subset G$ be a symmetric compact neighborhood of $e \in G$. Consider the compactly generated open subgroup $H = \bigcup_{n\geq 1} Q^n < G$. It further suffices to show that the map $H \to \mathscr{H}_\pi : g \mapsto \pi(g)\xi$ is continuous at $e \in H$. Thus, we may as well assume that G is σ-compact.

As usual, we denote by m_G a left invariant Haar measure on G. Let $\varepsilon > 0$ and set $B = \{g \in G \mid \|\pi(g)\xi - \xi\| < \varepsilon/2\}$. Then $B \subset G$ is a measurable subset since $B = \{g \in G \mid 2\Re(\langle \pi(g)\xi, \xi\rangle) > 2\|\xi\|^2 - \varepsilon^2/4\}$. Moreover, we have $B^{-1} = B$ and $B^2 = BB^{-1} \subset \{g \in G \mid \|\pi(g)\xi - \xi\| < \varepsilon\}$. Since $\pi(G)\xi \subset \mathscr{H}_\pi$ is separable, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $(\pi(g_n)\xi)_{n\in\mathbb{N}}$ is dense in $\pi(G)\xi$. This implies that $\bigcup_{n\in\mathbb{N}} g_n B = G$ and so $m_G(B) > 0$. Since G is σ-compact, up to replacing B by $B \cap K$ for a suitable symmetric compact subset, we may further assume that $B = B^{-1}$, $B \subset K$ and $0 < m_G(B) < +\infty$. Then $\mathbf{1}_B \in \mathrm{L}^2(G, \mathscr{B}(G), m_G)$ and $\varphi = \mathbf{1}_B * \mathbf{1}_B \in \mathrm{C}_c(G)$. Since $\varphi(e) = m_G(B) > 0$, the subset $U = \varphi^{-1}(0, +\infty)$ is open, $e \in U$ and $U \subset BB \subset \{g \in G \mid \|\pi(g)\xi - \xi\| < \varepsilon\}$. □

Definition 2.25. Let G be a locally compact group and $\pi : G \to \mathscr{U}(\mathscr{H}_\pi)$ a strongly continuous unitary representation. We say that

- π has *invariant vectors* and we write $1_G \subset \pi$ if the subspace of $\pi(G)$-invariant vectors
$$(\mathscr{H}_\pi)^G = \{\xi \in \mathscr{H}_\pi \mid \forall g \in G, \pi(g)\xi = \xi\}$$
is nonzero. Otherwise, we say that π is *ergodic* and we write $1_G \not\subset \pi$.
- π is *weakly mixing* if there exists a net $(g_i)_i$ in G such that $\pi(g_i) \to 0$ weakly as $g_i \to \infty$.

Whenever $\pi : G \to \mathscr{U}(\mathscr{H}_\pi)$ is a strongly continuous unitary representation, we consider the strongly continuous unitary representation $\pi \otimes \overline{\pi} : G \to \mathscr{U}(\mathscr{H} \otimes \overline{\mathscr{H}})$ defined by
$$\forall g \in G, \forall \xi, \eta \in \mathscr{H}, \quad (\pi \otimes \overline{\pi})(g)(\xi \otimes \eta) = \pi(g)\xi \otimes \overline{\pi(g)\eta}.$$
We prove the following useful characterization of weakly mixing unitary representations.

Proposition 2.26. *Let G be a locally compact group and $\pi : G \to \mathscr{U}(\mathscr{H}_\pi)$ a strongly continuous unitary representation. The following assertions are equivalent:*

(i) *π is weakly mixing.*
(ii) *π has no nonzero finite dimensional subrepresentation.*
(iii) *$\pi \otimes \overline{\pi}$ is ergodic.*

Proof. (i) $\Rightarrow$ (ii) By contraposition, assume that there is a nonzero finite dimensional subrepresentation $\rho \subset \pi$. Denote by $\mathscr{K} \subset \mathscr{H}$ the finite dimensional $\pi(G)$-invariant subspace associated with $\rho \subset \pi$. Let $(g_i)_i$ be a net in G. Since $\mathscr{K}$ is finite dimensional, the unitary group $\mathscr{U}(\mathscr{K})$ is compact and so there exist a subnet $(h_j)_j$ of $(g_i)_i$ and $v \in \mathscr{U}(\mathscr{K})$ such that $\rho(h_j) \to v$ in $\mathscr{U}(\mathscr{K})$ as $j \to \infty$. In particular, the net $(\pi(g_i))_i$ cannot converge to 0 weakly. Thus, π is not weakly mixing.

(ii) $\Rightarrow$ (iii) By contraposition, assume that there is a nonzero $(\pi\otimes\overline{\pi})(G)$-invariant vector $\zeta \in \mathscr{H} \otimes \overline{\mathscr{H}}$. Consider the nonzero Hilbert–Schmidt operator $W(\zeta) \in \mathrm{HS}(\mathscr{H})$ which satisfies $W(\zeta) \in \pi(G)'$. Set $T = W(\zeta)^* W(\zeta) \in \pi(G)'$ and note that $T^* = T$, $T \geq 0$ and $0 < \mathrm{Tr}(T) < +\infty$. Choose $\varepsilon > 0$ small enough so that the spectral projection $p = \mathbf{1}_{[\varepsilon, \varepsilon^{-1}]}(T) \in \pi(G)'$ is nonzero. Since $\varepsilon p \leq Tp \leq T$, we have $\mathrm{Tr}(p) \leq \varepsilon^{-1}\,\mathrm{Tr}(T) < +\infty$. Then $p(\mathscr{H}) \subset \mathscr{H}$ is a nonzero finite dimensional $\pi(G)$-invariant subspace. Thus, π has a nonzero finite dimensional subrepresentation.

(iii) $\Rightarrow$ (i) By contraposition, assume that π is not weakly mixing. Then there exist $\varepsilon > 0$ and a finite subset $\mathscr{F} \subset \mathscr{H}$ such that

$$\forall g \in G, \quad \sum_{\xi,\eta \in \mathscr{F}} |\langle \pi(g)\xi, \eta\rangle|^2 \geq \varepsilon.$$

Set $\zeta = \sum_{\xi \in \mathscr{F}} \xi \otimes \overline{\xi} \in \mathscr{H} \otimes \overline{\mathscr{H}}$. Then we have

$$\forall g \in G, \quad \langle (\pi(g) \otimes \overline{\pi(g)})\zeta, \zeta\rangle = \sum_{\xi,\eta \in \mathscr{F}} |\langle \pi(g)\xi, \eta\rangle|^2 \geq \varepsilon. \tag{2.1}$$

Consider the closed convex subset $\mathscr{C} = \overline{\mathrm{co}}\{(\pi(g)\otimes\overline{\pi(g)})\zeta \mid g \in G\} \subset \mathscr{H}\otimes\overline{\mathscr{H}}$ and denote by $c \in \mathscr{C}$ its unique circumcenter. Since $\mathscr{C}$ is $(\pi \otimes \overline{\pi})(G)$-invariant, it follows that $c \in \mathscr{C}$ is $(\pi \otimes \overline{\pi})(G)$-invariant. Moreover, (2.1) implies that $c \neq 0$. Thus, $\pi \otimes \overline{\pi}$ is not ergodic. □

For every $i \in \{1, 2\}$, let $\pi_i : G \to \mathscr{U}(\mathscr{H}_{\pi_i})$ be a strongly continuous unitary representation. We say that π_1 and π_2 are *unitarily equivalent* if there exists a unitary operator $U : \mathscr{H}_{\pi_1} \to \mathscr{H}_{\pi_2}$ such that for every $g \in G$, we have $\pi_2(g) = U\pi_1(g)U^*$. In this situation, we will identify π_1 with π_2.

2.2. Examples of unitary representations. Let G be a locally compact group.

The left regular representation λ_G. Let m_G be a left invariant Haar measure on G and simply denote by $\mathrm{L}^2(G) = \mathrm{L}^2(G, \mathscr{B}(G), m_G)$ the corresponding Hilbert space of L^2-integrable functions on G. Define the *left regular representation* $\lambda_G : G \to \mathscr{U}(\mathrm{L}^2(G))$ by the formula

$$\forall g \in G, \forall \xi \in \mathrm{L}^2(G), \quad (\lambda_G(g)\xi)(h) = \xi(g^{-1}h).$$

The left regular representation $\lambda_G : G \to \mathscr{U}(\mathrm{L}^2(G))$ is a strongly continuous unitary representation. This follows from the well known facts that the subspace $\mathrm{C}_c(G)$ of compactly supported continuous functions on G is $\|\cdot\|_2$-dense in $\mathrm{L}^2(G)$ and the left translation action $\lambda : G \curvearrowright \mathrm{C}_c(G)$ is $\|\cdot\|_\infty$-continuous (see Lemma 1.8).

PROPOSITION 2.27. *Keep the same notation as above. Then $1_G \subset \lambda_G$ if and only if G is compact.*

PROOF. If G is compact, then the left invariant Haar measure m_G is finite. This implies that the constant function $\mathbf{1}_G$ belongs to $\mathrm{L}^2(G)$ and is $\lambda_G(G)$-invariant. Conversely, assume that there exists a nonzero $\lambda_G(G)$-invariant vector $\xi \in \mathrm{L}^2(G)$.

CLAIM 2.28. There exists a σ-compact open subgroup $H < G$ such that $\xi = \mathbf{1}_H \xi$.

Indeed, define the measurable subsets $B = \{h \in G \mid \xi(h) \neq 0\}$ and $B_n = \{h \in G \mid |\xi(h)| \geq n^{-1}\}$ for every $n \geq 1$. Then $B = \bigcup_{n \geq 1} B_n$ and $m_G(B_n) < +\infty$ for every $n \geq 1$. By regularity, for every $n \geq 1$, there exists an open set $U_n \subset G$ such that $B_n \subset U_n$ and $m_G(U_n) < +\infty$. To prove the claim, it suffices to show that every open set $U \subset G$ with finite Haar measure is contained in a σ-compact open subgroup $H < G$.

Let $U \subset G$ be a nonempty open set such that $m_G(U) < +\infty$. Let $L < G$ be a σ-compact open subgroup. Since $m_G(U) < +\infty$, the set $\Lambda = \{gL \in G/L \mid U \cap gL \neq \emptyset\}$ is at most countable. Letting $H < G$ be the subgroup generated by L and Λ, we have that $U \subset H$ and $H < G$ is σ-compact and open. This finishes the proof of Claim 2.28.

Using Claim 2.28 and the assumption, for every $g \in G$, we have

$$\mathbf{1}_H \xi = \xi = \lambda_G(g)\xi = \lambda_G(g)(\mathbf{1}_H \xi) = \mathbf{1}_{gH}\xi = \mathbf{1}_{H \cap gH}\xi.$$

Since $\xi \neq 0$, we have $m_G(H \cap gH) > 0$ for every $g \in G$. It follows that $gH = H$ for every $g \in G$ and hence $H = G$. This shows that G is σ-compact.

We may now apply Fubini's theorem. Indeed, since for every $g \in G$ and m_G-almost every $h \in G$, we have $\xi(g^{-1}h) = \xi(h)$, Fubini's theorem implies that there exists $h \in G$ such that for m_G-almost every $g \in G$, we have $\xi(g^{-1}h) = \xi(h)$. This further implies that ξ is essentially constant. If we denote by $c > 0$ the essential value of $|\xi|^2$, we obtain $c \cdot m_G(G) = \|\xi\|^2 < +\infty$ and so $m_G(G) < +\infty$. Then G is compact by Proposition 1.6. $\square$

The Koopman representation κ. Let G be a locally compact second countable group and $(X, \mathscr{B}, \nu)$ a standard probability space. We simply write (X, ν) in what follows. We endow G with its σ-algebra $\mathscr{B}(G)$ of Borel subsets. Let $G \curvearrowright (X,\nu)$ be a *probability measure preserving* (pmp) action meaning that the action map $G \times X \to X : (g,x) \mapsto gx$ is measurable (where we endow $G \times X$ with the product σ-algebra $\mathscr{B}(G) \otimes \mathscr{B}$) and that $g_*\nu = \nu$ for every $g \in G$. Denote by $\mathrm{L}^2(X,\nu)$ the Hilbert space of L^2-integrable functions on X. Since (X,ν) is a standard probability space, $\mathrm{L}^2(X,\nu)$ is separable (see e.g. [**Zi84**, Theorem A.11]). Define the *Koopman representation* $\kappa : G \to \mathscr{U}(\mathrm{L}^2(X,\nu))$ associated with the pmp action $G \curvearrowright (X,\nu)$ by the formula

$$\forall g \in G, \forall \xi \in \mathrm{L}^2(X,\nu), \quad (\kappa(g)\xi)(x) = \xi(g^{-1}x).$$

The Koopman representation $\kappa : G \to \mathscr{U}(\mathrm{L}^2(X,\nu))$ is a strongly continuous unitary representation. This follows from Lemma 2.24 after noticing that for all $\xi, \eta \in \mathrm{L}^2(X,\nu)$, the map

$$\varphi_{\xi,\eta} : G \to \mathbb{C} : g \mapsto \langle \kappa(g)\xi, \eta\rangle = \int_X \xi(g^{-1}x)\overline{\eta(x)}\,\mathrm{d}\nu(x)$$

is measurable thanks to Fubini's theorem. The constant function $\mathbf{1}_X$ is $\kappa(G)$-invariant. For this reason, it is natural to consider the restriction of the Koopman representation to the orthogonal complement $\mathrm{L}^2(X,\nu)^0 = \mathrm{L}^2(X,\nu) \ominus \mathbb{C}\mathbf{1}_X$ that we denote by $\kappa^0 : G \to \mathscr{U}(\mathrm{L}^2(X,\nu)^0)$. By Proposition 2.15, we obtain the following useful characterization of ergodicity.

PROPOSITION 2.29. *Let $G \curvearrowright (X,\nu)$ be a pmp action. Then $G \curvearrowright (X,\nu)$ is ergodic if and only if $\kappa^0 : G \to \mathscr{U}(\mathrm{L}^2(X,\nu))$ is ergodic.*

Next, we say that the pmp action $G \curvearrowright (X,\nu)$ is *weakly mixing* if $\kappa^0 : G \to \mathscr{U}(\mathrm{L}^2(X,\nu))$ is weakly mixing. Using the Koopman representation, we obtain the following characterization of weakly mixing pmp actions.

PROPOSITION 2.30. *Let $G \curvearrowright (X,\nu)$ be a pmp action. The following assertions are equivalent:*

(i) *$G \curvearrowright (X,\nu)$ is weakly mixing.*
(ii) *$G \curvearrowright (X,\nu)$ is doubly ergodic.*
(iii) *$G \curvearrowright (X,\nu)$ is metrically ergodic.*

PROOF. The equivalence (i) $\Leftrightarrow$ (ii) follows by applying Proposition 2.26 to $\pi = \kappa^0$. By Proposition 2.22, we already know that (ii) $\Rightarrow$ (iii).

It remains to prove that (iii) $\Rightarrow$ (ii). Consider the separable metric space $(Z,d) = (\mathrm{L}^2(X,\nu), \|\cdot\|_2)$ and the continuous isometric action $\kappa : G \curvearrowright (\mathrm{L}^2(X,\nu), \|\cdot\|_2)$. Let $Y \subset X \times X$ be a nonnull G-invariant measurable subset. By Lemma 2.14(i), we may assume that Y is strictly G-invariant. For every $x \in X$, denote by $Y_x \subset X$ the measurable subset defined by $Y_x = \{y \in X \mid (x,y) \in Y\}$ and set $\xi_x = \mathbf{1}_{Y_x} \in \mathrm{L}^2(X,\nu)$. Then the map $f : X \to \mathrm{L}^2(X,\nu) : x \mapsto \xi_x$ is measurable and G-equivariant. Since $G \curvearrowright (X,\nu)$ is metrically ergodic, it follows that $f : X \to \mathrm{L}^2(X,\nu)$ is ν-almost everywhere constant. Choose $\xi \in \mathrm{L}^2(X,\nu)$ so that $\xi = \xi_x = \mathbf{1}_{Y_x}$ for ν-almost every $x \in X$. Since $(\nu \otimes \nu)(Y) > 0$, we have $\xi \neq 0$. Since $G \curvearrowright (X,\nu)$ is ergodic and since ξ is $\kappa(G)$-invariant, it follows that $\xi = \alpha \mathbf{1}_X$

for some $\alpha > 0$. This further implies that $\nu(Y_x) = 1$ for ν-almost every $x \in X$ and so $(\nu \otimes \nu)(Y) = 1$. Thus, $G \curvearrowright (X, \nu)$ is doubly ergodic. $\square$

The next proposition shows that ergodic pmp actions of locally compact groups satisfying the dynamical dichotomy for isometric actions are weakly mixing.

PROPOSITION 2.31. *Let G be a locally compact second countable group satisfying the dynamical dichotomy for isometric actions. Let $G \curvearrowright (X, \nu)$ be an ergodic pmp action. Then for any noncompact closed subgroup $H < G$, the action $H \curvearrowright (X, \nu)$ is weakly mixing.*

PROOF. Consider the separable metric space $(Z, d) = (\mathrm{L}^2(X, \nu)^0, \|\cdot\|_2)$ and the continuous isometric action $\kappa^0 : G \curvearrowright (\mathrm{L}^2(X, \nu)^0, \|\cdot\|_2)$. Since $G \curvearrowright (X, \nu)$ is ergodic, the action $G \curvearrowright Z$ has no global fixed point and so $G \curvearrowright Z$ is proper. Let $H < G$ be a closed subgroup and assume that $H \curvearrowright (X, \nu)$ is not weakly mixing. By Proposition 2.26, there exists a nonzero finite dimensional $\kappa^0(G)$-invariant subspace $\mathscr{K} \subset \mathrm{L}^2(X, \nu)^0$. Then $\mathrm{Ball}(\mathscr{K}) \subset Z$ is a H-invariant compact subset. Since the map $f : G \times Z \to Z \times Z$ is proper, $f^{-1}(\mathrm{Ball}(\mathscr{K}) \times \mathrm{Ball}(\mathscr{K}))$ is compact. Since $H \times \mathrm{Ball}(\mathscr{K}) \subset f^{-1}(\mathrm{Ball}(\mathscr{K}) \times \mathrm{Ball}(\mathscr{K}))$ is closed, it follows that $H < G$ is compact. $\square$

The quasi-regular representation $\lambda_{G/\Gamma}$. Let G be a locally compact second countable group and $\Gamma < G$ a lattice. We endow the locally compact second countable space $X = G/\Gamma$ with its σ-algebra $\mathscr{B}$ of Borel subsets (see Proposition 1.11(iii)). We denote by $\nu \in \mathrm{Prob}(X)$ the unique G-invariant Borel probability measure (see Proposition 1.15). Then the action $G \curvearrowright (X, \nu)$ is pmp. In that case, we denote by $\lambda_{G/\Gamma} : G \to \mathscr{U}(\mathrm{L}^2(G/\Gamma, \nu))$ the Koopman representation and we call it the *quasi-regular representation*. Since $G \curvearrowright X$ is transitive, Lemma 2.14 implies that $G \curvearrowright (X, \nu)$ is ergodic and Proposition 2.29 implies that $\lambda^0_{G/\Gamma} : G \to \mathscr{U}(\mathrm{L}^2(G/\Gamma, \nu)^0)$ is ergodic.

3. Amenability

3.1. Amenable groups.

DEFINITION 2.32. Let G be a locally compact group. We say that G is *amenable* if any affine continuous action $G \curvearrowright \mathscr{C}$ on a nonempty convex compact subset of a Hausdorff locally convex topological vector space has a G-fixed point.

We give a few examples of locally compact amenable groups.

PROPOSITION 2.33. *Any compact group is amenable.*

PROOF. Denote by m_G the (unique) Haar probability measure on G. Let $G \curvearrowright \mathscr{C}$ be an affine continuous action on a nonempty convex compact subset of a Hausdorff locally convex topological vector space. Define the convex weak*-compact subset $\mathrm{Prob}(\mathscr{C}) = \{\mu \in \mathrm{C}_{\mathbb{R}}(\mathscr{C})^* \mid \mu \geq 0 \text{ and } \mu(\mathbf{1}_{\mathscr{C}}) = 1\}$ and consider the affine weak*-continuous action $G \curvearrowright \mathrm{Prob}(\mathscr{C})$ defined by

$$\forall g \in G, \forall f \in \mathrm{C}_{\mathbb{R}}(\mathscr{C}), \forall \mu \in \mathrm{Prob}(\mathscr{C}), \quad (g_*\mu)(f) = \mu(f \circ g).$$

Define the *barycenter* map $\mathrm{Bar} : \mathrm{Prob}(\mathscr{C}) \to \mathscr{C}$ as the unique continuous map satisfying $f(\mathrm{Bar}(\mu)) = \mu(f)$ for every real-valued continuous affine function $f \in \mathscr{A}_{\mathbb{R}}(\mathscr{C})$. Since $G \curvearrowright \mathscr{C}$ is continuous affine, $\mathrm{Bar} : \mathrm{Prob}(\mathscr{C}) \to \mathscr{C}$ is G-equivariant. Choose a point $c \in \mathscr{C}$ and define the G-equivariant continuous orbital map $\iota : G \to \mathscr{C} : g \mapsto gc$. We may define $\mu = \iota_* m_G \in \mathrm{Prob}(\mathscr{C})$. Since m_G is a left invariant Borel measure, it follows that $g_*\mu = \mu$ for every $g \in G$. This further implies that $\mathrm{Bar}(\mu) \in \mathscr{C}$ is a G-fixed point. □

PROPOSITION 2.34. *Any abelian locally compact group is amenable.*

PROOF. Let $G \curvearrowright \mathscr{C}$ be an affine continuous action on a nonempty convex compact subset of a Hausdorff locally convex topological vector space. Whenever $\mathscr{F} \subset G$ is a finite subset, denote by $\mathscr{C}^{\mathscr{F}}$ the convex compact subset of $\mathscr{F}$-fixed points in $\mathscr{C}$. Since G is abelian, G leaves $\mathscr{C}^{\mathscr{F}}$ globally invariant. If we show that the compact subset $\mathscr{C}^{\mathscr{F}}$ is nonempty for every finite subset $\mathscr{F} \subset G$, by finite intersection property, we will have that the compact subset of G-fixed points $\mathscr{C}^G = \bigcap\{\mathscr{C}^{\mathscr{F}} \mid \mathscr{F} \subset G \text{ finite subset}\}$ is nonempty. It remains to prove that $\mathscr{C}^{\mathscr{F}}$ is nonempty for every finite subset $\mathscr{F} \subset G$. By induction and since G is abelian, it suffices to prove that $\mathscr{C}^g = \{c \in \mathscr{C} \mid gc = c\}$ is nonempty for every $g \in G$. This in turn follows from Markov–Kakutani's fixed point theorem. Choose $c \in \mathscr{C}$ and for every $n \in \mathbb{N}$, set

$$c_n = \frac{1}{n+1}(c + gc + \cdots + g^n c) \in \mathscr{C}.$$

By compactness, denote by $c_\infty \in \mathscr{C}$ an accumulation point of the sequence $(c_n)_{n \in \mathbb{N}}$. Since $\frac{1}{n+2}c + \frac{n+1}{n+2}gc_n = \frac{n+1}{n+2}c_n + \frac{1}{n+2}g^{n+1}c$ and since g is a homeomorphism of $\mathscr{C}$, it follows that $gc_\infty = c_\infty$ and so $c_\infty \in \mathscr{C}^g$. □

We prove various permanence properties enjoyed by amenable locally compact groups.

PROPOSITION 2.35. *Let G, H be locally compact groups. Assume that G is amenable. The following assertions hold:*

(i) *If $\rho : G \to H$ is a continuous homomorphism with dense range, then H is amenable.*
(ii) *If $H \lhd G$ is a closed normal subgroup, then G/H is amenable.*

PROOF. (i) Let $H \curvearrowright \mathscr{C}$ be an affine continuous action on a nonempty convex compact subset of a Hausdorff locally convex topological vector space. By composing with $\rho : G \to H$, we obtain an affine continuous G-action. Since G is amenable, the affine continuous G-action has a G-fixed point. This shows that the original affine continuous H-action has a $\rho(G)$-fixed point. By continuity and density of $\rho(G)$ in H, we obtain a H-fixed point. Thus, $H = \overline{\rho(G)}$ is amenable.

(ii) It suffices to apply item (i) to the continuous homomorphism $G \to G/H$. □

Let now G be a locally compact σ-compact group. As usual, we denote by $\mathscr{B}(G)$ the σ-algebra of Borel subsets of G and we fix a left invariant Haar measure m_G on G. Denote by $\Delta_G : G \to \mathbb{R}^*_+$ the modular function. For every $p \in [1, +\infty]$, we simply write $\mathrm{L}^p(G) = \mathrm{L}^p(G, \mathscr{B}(G), m_G)$. Since G is σ-compact, m_G is σ-finite and hence we have $\mathrm{L}^\infty(G) = \mathrm{L}^1(G)^*$. We denote by $\lambda : G \curvearrowright \mathrm{L}^p(G)$ the *left translation action* defined by

$$\forall g \in G, \forall F \in \mathrm{L}^p(G), \quad (\lambda(g)F)(h) = F(g^{-1}h).$$

The left translation action $\lambda : G \curvearrowright \mathrm{L}^p(G)$ is isometric for every $p \in [1, +\infty]$ and continuous for every $p \in [1, +\infty)$. Since $G \curvearrowright \mathrm{L}^\infty(G)$ need not be continuous, we denote by $\mathrm{UC}_\ell(G) \subset \mathrm{L}^\infty(G)$ the subspace of *left uniformly continuous* functions

$$\mathrm{UC}_\ell(G) = \{F \in \mathrm{L}^\infty(G) \mid \|\lambda(g)F - F\|_\infty \to 0 \text{ as } g \to e\}.$$

Observe that $\mathrm{UC}_\ell(G) \subset \mathrm{L}^\infty(G)$ is a $\lambda(G)$-invariant $\|\cdot\|_\infty$-closed subspace. Letting $\mathrm{C}_b(G)$ be the space of bounded continuous functions on G, we have the following inclusions $\mathrm{UC}_\ell(G) \subset \mathrm{C}_b(G) \subset \mathrm{L}^\infty(G)$. Observe that when G is discrete, we have $\mathrm{UC}_\ell(G) = \mathrm{C}_b(G) = \ell^\infty(G)$. Whenever $\mathscr{F} \subset \mathrm{L}^\infty(G)$ is a $\|\cdot\|_\infty$-closed subspace such that $\mathbb{C}\mathbf{1}_G \subset \mathscr{F}$, we say that an element $\mathfrak{m} \in \mathscr{F}^*$ is a *mean* if $\mathfrak{m}(F) \geq 0$ for every $F \in \mathscr{F}_+$ and $\mathfrak{m}(\mathbf{1}_G) = 1$. If $\mathscr{F} \subset \mathrm{L}^\infty(G)$ is moreover $\lambda(G)$-invariant, we say that $\mathfrak{m} \in \mathscr{F}^*$ is a *left invariant mean* if $\mathfrak{m}(\lambda(g)F) = \mathfrak{m}(F)$ for every $g \in G$ and every $F \in \mathscr{F}$.

Recall that the *convolution product* of two measurable functions $F_1, F_2 : G \to \mathbb{C}$, whenever it makes sense, is defined as

$$(F_1 * F_2)(h) = \int_G F_1(g) F_2(g^{-1}h)\, \mathrm{d}m_G(g).$$

Set $\mathrm{P}(G) = \{\mu \in \mathrm{L}^1(G) \mid \mu \geq 0 \text{ and } \|\mu\|_1 = 1\}$. We will use the following technical lemma whose proof is left to the reader.

Lemma 2.36. *The following assertions hold:*

(i) *If* $\mu \in \mathrm{P}(G)$ *and* $F \in \mathrm{L}^\infty(G)$, *then* $\mu * F \in \mathrm{UC}_\ell(G)$.
(ii) *If* $(\mu_i)_{i \in I}$ *is a net in* $\mathrm{L}^1(G)$ *such that* $\lim_i \|\mu_i\|_1 = 0$, *then for every* $F \in \mathrm{L}^\infty(G)$, *we have* $\lim_i \|\mu_i * F\|_\infty = 0$.
(iii) *There exists a net* $(\mu_i)_{i \in I}$ *in* $\mathrm{P}(G)$ *such that for every* $\mu \in \mathrm{L}^1(G)$, *we have* $\lim_i \|\mu_i * \mu - \mu\|_1 = \lim_i \|\mu * \mu_i - \mu\|_1 = 0$.
(iv) *If* $g \in G$, $\mu \in \mathrm{P}(G)$ *and* $F \in \mathrm{L}^\infty(G)$, *then* $(\lambda(g)\mu) * F = \lambda(g)(\mu * F)$.

The main result of this section is a functional analytic characterization of amenability for locally compact groups.

Theorem 2.37. *Let G be a locally compact σ-compact group. The following conditions are equivalent:*

(i) $1_G \prec \lambda_G$, *that is, the left regular representation* λ_G *has almost invariant vectors.*

(ii) *There exists a left invariant mean* $\mathfrak{m} \in \mathrm{L}^\infty(G)^*$.
(iii) *There exists a left invariant mean* $\mathfrak{m} \in \mathrm{UC}_\ell(G)^*$.
(iv) G *is amenable, that is, any affine continuous action* $G \curvearrowright \mathscr{C}$ *on a nonempty convex compact subset of a Hausdorff locally convex topological vector space has a* G*-fixed point.*

Proof. (i) $\Rightarrow$ (ii) There exists a net $(\xi_i)_{i \in I}$ of unit vectors in $\mathrm{L}^2(G)$ such that for every compact subset $Q \subset G$, we have

$$\lim_i \sup_{g \in Q} \|\lambda_G(g)\xi_i - \xi_i\|_2 = 0.$$

Choose a nonprincipal ultrafilter $\mathcal{U}$ on I. Define the unital $*$-homomorphism $\rho : \mathrm{L}^\infty(G) \to \mathrm{B}(\mathrm{L}^2(G))$ by the formula $\rho(F)\xi = F\xi$ for every $F \in \mathrm{L}^\infty(G)$ and every $\xi \in \mathrm{L}^2(G)$. Then we have $\lambda_G(g)\rho(F)\lambda_G(g)^* = \rho(\lambda(g)F)$ for every $g \in G$ and every $F \in \mathrm{L}^\infty(G)$. Define the mean $\mathfrak{m} \in \mathrm{L}^\infty(G)^*$ by the formula

$$\forall F \in \mathrm{L}^\infty(G), \quad \mathfrak{m}(F) = \lim_{i \to \mathcal{U}} \langle \rho(F)\xi_i, \xi_i \rangle.$$

Then for every $g \in G$ and every $F \in \mathrm{L}^\infty(G)$, we have

$$\begin{aligned} \mathfrak{m}(\lambda(g)F) &= \lim_{i \to \mathcal{U}} \langle \rho(\lambda(g)F)\xi_i, \xi_i \rangle \\ &= \lim_{i \to \mathcal{U}} \langle \lambda_G(g)\rho(F)\lambda_G(g)^*\xi_i, \xi_i \rangle \\ &= \lim_{i \to \mathcal{U}} \langle \rho(F)\lambda_G(g)^*\xi_i, \lambda_G(g)^*\xi_i \rangle \\ &= \mathfrak{m}(F). \end{aligned}$$

Thus, $\mathfrak{m} \in \mathrm{L}^\infty(G)^*$ is a left invariant mean.

(ii) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (iv) As in Proposition 2.33, define the convex weak*-compact subset $\mathrm{Prob}(\mathscr{C}) = \{\mu \in \mathrm{C}_\mathbb{R}(\mathscr{C})^* \mid \mu \geq 0 \text{ and } \mu(\mathbf{1}_\mathscr{C}) = 1\}$ and consider the affine weak*-continuous action $G \curvearrowright \mathrm{Prob}(\mathscr{C})$ defined by

$$\forall g \in G, \forall f \in \mathrm{C}_\mathbb{R}(\mathscr{C}), \forall \mu \in \mathrm{Prob}(\mathscr{C}), \quad (g_*\mu)(f) = \mu(f \circ g).$$

Recall that the barycenter map $\mathrm{Bar} : \mathrm{Prob}(\mathscr{C}) \to \mathscr{C}$ is the unique continuous map satisfying $f(\mathrm{Bar}(\mu)) = \mu(f)$ for every real-valued continuous affine function $f \in \mathscr{A}_\mathbb{R}(\mathscr{C})$. Since $G \curvearrowright \mathscr{C}$ is continuous affine, $\mathrm{Bar} : \mathrm{Prob}(\mathscr{C}) \to \mathscr{C}$ is G-equivariant. Choose a point $c \in \mathscr{C}$ and define the G-equivariant continuous orbital map $\iota : G \to \mathscr{C} : g \mapsto gc$. For every $f \in \mathrm{C}_\mathbb{R}(\mathscr{C})$, we have $f \circ \iota \in \mathrm{UC}_\ell(G)$. We may define $\mu \in \mathrm{Prob}(\mathscr{C})$ by the formula

$$\forall f \in \mathrm{C}_\mathbb{R}(\mathscr{C}), \quad \mu(f) = \mathfrak{m}(f \circ \iota).$$

Since $\mathfrak{m} \in \mathrm{UC}_\ell(G)^*$ is a left invariant mean, it follows that $g_*\mu = \mu$ for every $g \in G$. This further implies that $\mathrm{Bar}(\mu) \in \mathscr{C}$ is a G-fixed point.

(iv) $\Rightarrow$ (iii) Endow $E = \mathrm{UC}_\ell(G)^*$ with the weak*-topology and consider the nonempty convex weak*-compact subset $\mathscr{C} \subset \mathrm{UC}_\ell(G)^*$ of all means on $\mathrm{UC}_\ell(G)$. Since the action $G \curvearrowright \mathrm{UC}_\ell(G)$ is $\| \cdot \|_\infty$-continuous, the action $G \curvearrowright \mathscr{C}$ is affine weak*-continuous. Thus, there exists a G-fixed point $\mathfrak{m} \in \mathscr{C}$ and so $\mathfrak{m} \in \mathrm{UC}_\ell(G)^*$ is a left invariant mean.

(iii) $\Rightarrow$ (i) We proceed in several intermediate steps. Let $\mathfrak{m} \in \mathrm{UC}_\ell(G)^*$ be a left invariant mean.

Claim 2.38. For every $\mu \in \mathrm{P}(G)$ and every $F \in \mathrm{UC}_\ell(G)$, we have $\mathfrak{m}(\mu * F) = \mathfrak{m}(F)$.

Indeed, let $\mu \in \mathrm{P}(G)$ and $F \in \mathrm{UC}_\ell(G)$. Observe that using Lemma 2.36(ii), we may assume that $\mu \in \mathrm{P}(G)$ is compactly supported. Then denote by $K = \mathrm{supp}(\mu) \subset G$ the compact support of $\mu \in \mathrm{P}(G)$. The G-equivariant mapping $\iota : G \to \mathrm{UC}_\ell(G) : g \mapsto \lambda(g)F$ is continuous and thus $\iota(K) \subset \mathrm{UC}_\ell(G)$ is a compact subset. Then the closed convex hull $\mathscr{C}$ of $\iota(K)$ is a convex compact subset of $\mathrm{UC}_\ell(G)$ (see [**Ru91**, Theorem 3.20]). Set $\nu = \iota_*\mu$ and regard $\nu \in \mathrm{Prob}(\mathscr{C})$ by the formula

$$\forall f \in \mathrm{C}_\mathbb{R}(\mathscr{C}), \quad \nu(f) = \int_G \mu(g) f(\lambda(g)F)\, \mathrm{d}m_G(g).$$

We claim that $\mu * F = \mathrm{Bar}(\nu) \in \mathscr{C}$. Recall that $f(\mathrm{Bar}(\nu)) = \nu(f)$ for every $f \in \mathscr{A}_\mathbb{R}(\mathscr{C})$. For every $h \in G$, regarding the evaluation map $e_h : \mathrm{UC}_\ell(G) \to \mathbb{C} : f \mapsto f(h)$ as an element of $\mathscr{A}_\mathbb{R}(\mathscr{C})$, we have

$$\mathrm{Bar}(\nu)(h) = e_h(\mathrm{Bar}(\nu)) = \nu(e_h) = \int_G \mu(g) e_h(\lambda(g)F)\, \mathrm{d}m_G(g) = (\mu * F)(h).$$

Thus, we have $\mathrm{Bar}(\nu) = \mu * F$. Since $\mathfrak{m} \in \mathrm{UC}_\ell(G)^*$ is a left invariant mean, we can regard $\mathfrak{m} \in \mathscr{A}_\mathbb{R}(\mathscr{C})$ and we obtain

$$\mathfrak{m}(\mu * F) = \mathfrak{m}(\mathrm{Bar}(\nu)) = \int_G \mu(g)\mathfrak{m}(\lambda(g)F)\, \mathrm{d}m_G(g) = \mathfrak{m}(F).$$

This finishes the proof of Claim 2.38.

Claim 2.39. There exists a mean $\mathfrak{m}_0 \in \mathrm{L}^\infty(G)^*$ such that for every $\mu \in \mathrm{P}(G)$ and every $F \in \mathrm{L}^\infty(G)$, we have $\mathfrak{m}_0(\mu * F) = \mathfrak{m}_0(F)$.

Indeed, choose $\mu_0 \in \mathrm{P}(G)$. Thanks to Lemma 2.36(i), we may define the mean $\mathfrak{m}_0 \in \mathrm{L}^\infty(G)^*$ by the formula $\mathfrak{m}_0(F) = \mathfrak{m}(\mu_0 * F)$ for every $F \in \mathrm{L}^\infty(G)$. Choose a net as in Lemma 2.36(iii). Using Lemma 2.36(ii), for every $\mu \in \mathrm{P}(G)$, we have

$$\begin{aligned}
\mathfrak{m}_0(\mu * F) &= \lim_i \mathfrak{m}_0(\mu * \mu_i * F) \\
&= \lim_i \mathfrak{m}(\mu_0 * \mu * \mu_i * F) \\
&= \lim_i \mathfrak{m}(\mu_i * F) \quad \text{by Claim 2.38} \\
&= \lim_i \mathfrak{m}(\mu_0 * \mu_i * F) \quad \text{by Claim 2.38} \\
&= \mathfrak{m}(\mu_0 * F) \\
&= \mathfrak{m}_0(F).
\end{aligned}$$

This finishes the proof of Claim 2.39.

Denote by $\mathscr{M}$ the nonempty convex weak*-compact subset of all means on $\mathrm{L}^\infty(G)$. Hahn–Banach theorem implies that the map $\mathrm{P}(G) \to \mathscr{M}$:

$\mu \mapsto \mathfrak{m}_\mu$ defined by the formula $\mathfrak{m}_\mu(F) = \int_G \mu(g)F(g)\,\mathrm{d}m_G(g)$ for every $F \in \mathrm{L}^\infty(G)$ has dense range. Thus, we can find a net $(\mu_i)_{i\in I}$ in $\mathrm{P}(G)$ such that $\mathfrak{m}_{\mu_i} \to \mathfrak{m}_0$ for the weak*-topology. For every $\mu \in \mathrm{P}(G)$, define $\mu^{\mathrm{op}} \in \mathrm{P}(G)$ by the formula $\mu^{\mathrm{op}}(g) = \Delta_G(g)^{-1}\mu(g^{-1})$. For every $\mu \in \mathrm{P}(G)$ and every $F \in \mathrm{L}^\infty(G)$, using Fubini's theorem, we have

$$\begin{aligned}\int_G (\mu * \mu_i)(g)F(g)\,\mathrm{d}m_G(g) &= \int_{G\times G} \mu(h)\mu_i(h^{-1}g)\,F(g)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\ &= \int_{G\times G} \mu_i(h^{-1}g)\,\mu(h)F(g)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\ &= \int_{G\times G} \mu_i(g)\,\mu(h)F(hg)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\ &= \int_{G\times G} \mu_i(g)\,\mu^{\mathrm{op}}(h)F(h^{-1}g)\,\mathrm{d}m_G^{\otimes 2}(g,h)\\ &= \int_{G\times G} \mu_i(g)\,(\mu^{\mathrm{op}} * F)(g)\,\mathrm{d}m_G(g).\end{aligned}$$

Then Claim 2.39 implies that for every $\mu \in \mathrm{P}(G)$, $\mu * \mu_i - \mu_i \to 0$ weakly in $\mathrm{L}^1(G)$. Denote by J the directed set of all pairs $(\varepsilon, \mathscr{F})$ where $\varepsilon > 0$ and $\mathscr{F} \subset \mathrm{P}(G)$ is a finite subset endowed with the order $(\varepsilon_1, \mathscr{F}_1) \leq (\varepsilon_2, \mathscr{F}_2)$ if and only if $\varepsilon_1 \leq \varepsilon_2$ and $\mathscr{F}_2 \subset \mathscr{F}_1$. Let $j = (\varepsilon, \mathscr{F}) \in J$ and consider the Banach space $(E_j, \|\cdot\|) = \bigoplus_{\mu\in\mathscr{F}}(\mathrm{L}^1(G), \|\cdot\|_1)$. The weak topology on E_j is simply the product of the weak topologies on $\mathrm{L}^1(G)$. Then 0 belongs to the weak closure in E_j of the convex subset

$$\mathscr{C}_j = \{(\mu * \psi - \psi)_{\mu\in\mathscr{F}} \mid \psi \in \mathrm{P}(G)\} \subset E_j.$$

Hahn–Banach theorem implies that 0 belongs to the strong closure in E_j of $\mathscr{C}_j$. Then we may find $\psi_j \in \mathrm{P}(G)$ such that for every $\mu \in \mathscr{F}$, we have $\|\mu * \psi_j - \psi_j\|_1 < \varepsilon$. Thus, we have found a net $(\psi_j)_{j\in J}$ in $\mathrm{P}(G)$ such that for every $\mu \in \mathrm{P}(G)$, we have $\lim_j \|\mu * \psi_j - \psi_j\|_1 = 0$.

Note that for every nonempty $\|\cdot\|_1$-compact subset $K \subset \mathrm{P}(G)$, we have $\lim_j \|\mu * \psi_j - \psi_j\|_1 = 0$ *uniformly* on K. Indeed, let $\varepsilon > 0$ and choose $\mu_1, \dots, \mu_n \in K$ such that for every $\mu \in K$, there exists $1 \leq i \leq n$ for which $\|\mu - \mu_i\| \leq \varepsilon$. Choose $j_0 \in J$ such that $\|\mu_i * \psi_j - \psi_j\|_1 \leq \varepsilon$ for every $1 \leq i \leq n$ and every $j \geq j_0$. Then for every $\mu \in K$ and every $j \geq j_0$, choosing $1 \leq i \leq n$ such that $\|\mu - \mu_i\| \leq \varepsilon$, we have

$$\begin{aligned}\|\mu * \psi_j - \psi_j\|_1 &\leq \|(\mu - \mu_i) * \psi_j\|_1 + \|\mu_i * \psi_j - \psi_j\|_1\\ &\leq \|\mu - \mu_i\|_1 + \|\mu_i * \psi_j - \psi_j\|_1\\ &\leq 2\varepsilon.\end{aligned}$$

This shows that $\lim_j \|\mu * \psi_j - \psi_j\|_1 = 0$ *uniformly* on K.

Fix $\varepsilon > 0$ and $Q \subset G$ a compact subset. Fix $\mu \in \mathrm{P}(G)$. The orbital map $G \to \mathrm{P}(G) : g \mapsto \lambda(g)\mu$ is $\|\cdot\|_1$-continuous and so $\iota(Q) \subset \mathrm{P}(G)$ is

$\|\cdot\|_1$-compact. Lemma 2.36(iv) implies that

$$\sup_{g\in Q}\|\lambda(g)(\mu*\psi_j)-\mu*\psi_j\|_1=\sup_{g\in Q}\|(\lambda(g)\mu)*\psi_j-\mu*\psi_j\|_1\to 0.$$

We may find $j\in J$ large enough so that with $\zeta=\mu*\psi_j\in \mathrm{P}(G)$, we have

$$\sup_{g\in Q}\|\lambda(g)\zeta-\zeta\|_1\leq\varepsilon^2.$$

Set $\xi=\zeta^{1/2}\in \mathrm{L}^2(G)_+$ and observe that $\|\xi\|=1$. Moreover, we have

$$\begin{aligned}\sup_{g\in Q}\|\lambda_G(g)\xi-\xi\|_2^2&=\sup_{g\in Q}\int_G|\xi(g^{-1}h)-\xi(h)|^2\,\mathrm{d}m_G(h)\\&=\sup_{g\in Q}\int_G|\zeta(g^{-1}h)^{1/2}-\zeta(h)^{1/2}|^2\,\mathrm{d}m_G(h)\\&\leq\sup_{g\in Q}\int_G|\zeta(g^{-1}h)-\zeta(h)|\,\mathrm{d}m_G(h)\\&=\sup_{g\in Q}\|\lambda(g)\zeta-\zeta\|_1\leq\varepsilon^2.\end{aligned}$$

This implies that $1_G\prec\lambda_G$ and finishes the proof of Theorem 2.37. ☐

We conclude this section by proving von Neumann's result regarding nonamenability of free groups.

Theorem 2.40 (von Neumann). *Denote by* $\mathbf{F}_2=\langle a,b\rangle$ *the free group on two generators. Then* $\mathbf{F}_2$ *is nonamenable.*

Proof. By contradiction, assume that $\mathbf{F}_2=\langle a,b\rangle$ is amenable. Denote by $\mathfrak{m}\in\ell^\infty(\mathbf{F}_2)^*$ a left invariant mean. Define $\mathfrak{n}:\mathscr{P}(\mathbf{F}_2)\to[0,1]:W\mapsto\mathfrak{m}(\mathbf{1}_W)$ and observe that $\mathfrak{n}$ is a finitely additive left invariant *probability mean* on $\mathbf{F}_2$. Then we necessarily have $\mathfrak{n}(F)=0$ for every finite subset $F\subset\mathbf{F}_2$. In particular, we have $\mathfrak{n}(\{e\})=0$.

Denote by $W_a\subset\mathbf{F}_2$ the subset of reduced words whose first letter is a. Likewise, consider the subsets $W_{a^{-1}},W_b,W_{b^{-1}}\subset\mathbf{F}_2$. Observe that $\mathbf{F}_2\setminus\{e\}=W_a\sqcup W_{a^{-1}}\sqcup W_b\sqcup W_{b^{-1}}$. Since $a\cdot(W_a\sqcup W_b\sqcup W_{b^{-1}})\subset W_a$, it follows that

$$\begin{aligned}\mathfrak{n}(W_a)+\mathfrak{n}(W_b)+\mathfrak{n}(W_{b^{-1}})&=\mathfrak{n}(W_a\sqcup W_b\sqcup W_{b^{-1}})\\&=\mathfrak{n}(a\cdot(W_a\sqcup W_b\sqcup W_{b^{-1}}))\\&\leq\mathfrak{n}(W_a).\end{aligned}$$

This implies that $\mathfrak{n}(W_b)=\mathfrak{n}(W_{b^{-1}})=0$. Likewise, we have $\mathfrak{n}(W_a)=\mathfrak{n}(W_{a^{-1}})=0$. This further implies that $\mathfrak{n}(\mathbf{F}_2)=0$, a contradiction. ☐

One can show that amenability is inherited by closed subgroups (see e.g. [**Zi84**, Proposition 4.2.20]). Thus, any locally compact group that contains $\mathbf{F}_2$ as a closed subgroup is nonamenable.

3.2. Amenable actions. For every $p \in [1, +\infty]$, we simply denote by $\mathrm{L}^p(G) = \mathrm{L}^p(G, \mathscr{B}(G), m_G)$ and by $\lambda : G \curvearrowright \mathrm{L}^p(G)$ the left translation action. Let $G \curvearrowright (X, \nu)$ be a nonsingular action and denote by $\sigma : G \curvearrowright \mathrm{L}^\infty(X)$ the corresponding weak*-continuous action. Simply write $\mathrm{L}^\infty(G \times X) = \mathrm{L}^\infty(G \times X, m_G \otimes \nu)$. Denote by $\lambda \otimes \sigma : G \curvearrowright \mathrm{L}^\infty(G \times X)$ the weak*-continuous action arising from the diagonal nonsingular action $G \curvearrowright (G \times X, m_G \otimes \nu)$.

Definition 2.41. We say that a nonsingular action $G \curvearrowright (X, \nu)$ is *amenable* if there exists a unital positive linear contractive mapping $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ such that

(i) For every $f \in \mathrm{L}^\infty(X)$, we have $\Phi(\mathbf{1}_G \otimes f) = f$.
(ii) For every $g \in G$ and every $F \in \mathrm{L}^\infty(G \times X)$, we have

$$\Phi((\lambda \otimes \sigma)(g)F) = \sigma(g)\Phi(F).$$

We simply say that $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ is a *G-equivariant projection.*

Amenable actions are very useful as they provide the existence of equivariant measurable maps.

Theorem 2.42. *Let G be a locally compact second countable group and $G \curvearrowright (X, \nu)$ an amenable nonsingular action. Let Y be a compact metrizable space and $G \curvearrowright Y$ a continuous action. Then there exists a G-equivariant measurable map $\beta : X \to \mathrm{Prob}(Y)$.*

Proof. Denote by $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ the G-equivariant projection witnessing that the nonsingular action $G \curvearrowright (X, \nu)$ is amenable. Choose a point $y \in Y$. Consider the G-equivariant unital positive linear contractive mapping $\Psi : \mathrm{C}(Y) \mapsto \mathrm{L}^\infty(G) : f \mapsto (g \mapsto f(gy))$. Regard $\mathrm{L}^\infty(G) \subset \mathrm{L}^\infty(G \times X)$ and define the G-equivariant unital positive linear contractive mapping $\Theta = \Phi \circ \Psi : \mathrm{C}(Y) \to \mathrm{L}^\infty(X)$.

Since Y is compact metrizable, $\mathrm{C}(Y)$ is $\|\cdot\|_\infty$-separable. We may choose a countable $\|\cdot\|_\infty$-dense subset $\mathscr{S} \subset \mathrm{C}(Y)_+$ such that $\mathbf{1}_Y \in \mathscr{S}$. Denote by $\mathscr{D} \subset \mathrm{C}(Y)$ the countable $\|\cdot\|_\infty$-dense $\mathbb{Q}[i]$-linear subspace generated by $\mathscr{S}$. Then we may choose a conull measurable subset $X_0 \subset X$ such that $\Theta|_{\mathscr{D}} : \mathscr{D} \to \mathrm{L}^\infty(X)$ induces a unital positive $\mathbb{Q}[i]$-linear contractive mapping $\Theta_{0,\mathscr{D}} : \mathscr{D} \to \mathscr{L}^\infty(X_0)$. By $\|\cdot\|_\infty$-density of $\mathscr{D}$ in $\mathrm{C}(Y)$, we may uniquely extend $\Theta_{0,\mathscr{D}}$ to a unital positive linear contractive mapping $\Theta_0 : \mathrm{C}(Y) \to \mathscr{L}^\infty(X_0)$. Observe that for every $f \in \mathrm{C}(Y)$, the class of $\Theta_0(f)$ in $\mathrm{L}^\infty(X)$ is equal to $\Theta(f) \in \mathrm{L}^\infty(X)$. Using Riesz's representation theorem, we obtain a measurable map $\beta^0 : X_0 \to \mathrm{Prob}(Y)$ such that for every $x \in X_0$ and every $f \in \mathrm{C}(Y)$, we have $\beta^0_x(f) = \Theta_0(f)(x)$. We may extend β^0 to a measurable map $\beta : X \to \mathrm{Prob}(Y)$ by letting $\beta_x = \eta \in \mathrm{Prob}(Y)$ for every $x \in X \setminus X_0$, where $\eta \in \mathrm{Prob}(Y)$ is some Borel probability measure on Y.

It remains to check that $\beta : X \to \mathrm{Prob}(Y)$ is G-equivariant. Fix $g \in G$. Let $f \in \mathrm{C}(Y)$. Then for every $x \in X_0 \cap g^{-1}X_0$, we have $\beta_{gx}(f) = \Theta_0(f)(gx)$ and $(g_*\beta_x)(f) = \Theta_0(f \circ g)(x)$. Since $\Theta(f \circ g) = \Theta(f) \circ g$ in $\mathrm{L}^\infty(X)$, it follows that $\beta_{gx}(f) = (g_*\beta_x)(f)$ for ν-almost every $x \in X$. Considering the countable $\|\cdot\|_\infty$-dense subset $\mathscr{D} \subset \mathrm{C}(Y)$, there exists a conull measurable subset $X_1 \subset X$ such that $\beta_{gx}(f) = (g_*\beta_x)(f)$ for every $x \in X_1$ and every $f \in \mathscr{D}$. By $\|\cdot\|_\infty$-density of $\mathscr{D} \subset \mathrm{C}(Y)$, we obtain $\beta_{gx}(f) = (g_*\beta_x)(f)$ for every $x \in X_1$ and every $f \in \mathrm{C}(Y)$. This implies that $\beta_{gx} = g_*\beta_x$ for every $x \in X_1$. Thus, the measurable map $\beta : X \to \mathrm{Prob}(Y)$ is G-equivariant. $\square$

Recall that $\mathrm{P}(G) = \{\mu \in \mathrm{L}^1(G) \mid \mu \geq 0 \text{ and } \|\mu\|_1 = 1\}$. For every $\mu \in \mathrm{L}^1(G)$ and every $F \in \mathrm{L}^\infty(G \times X)$, we denote by $(\mu \otimes \mathrm{id}_X)(F) \in \mathrm{L}^\infty(X)$ the unique element that satisfies

$$\forall \psi \in \mathrm{L}^1(X, \nu), \quad \psi((\mu \otimes \mathrm{id}_X)(F)) = (\mu \otimes \psi)(F).$$

If $\mu \in \mathrm{P}(G)$, then $\mu \otimes \mathrm{id}_X : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ is a unital positive linear contractive mapping. If $(\mu_i)_{i\in I}$ is a net in $\mathrm{L}^1(G)$ such that $\lim_i \|\mu_i\|_1 = 0$, then for every $F \in \mathrm{L}^\infty(G \times X)$, we have $(\mu_i \otimes \mathrm{id}_X)(F) \to 0$ with respect to the weak*-topology.

Firstly, we observe that all nonsingular actions of amenable groups are amenable.

PROPOSITION 2.43. *Let G be an amenable locally compact second countable group. Then any nonsingular action $G \curvearrowright (X, \nu)$ is amenable.*

PROOF. Since G is amenable, there exists a net of elements $(\mu_i)_{i\in I}$ in $\mathrm{P}(G)$ such that $\|\lambda(g)\mu_i - \mu_i\|_1 \to 0$ uniformly on compact subsets $K \subset G$ (see the proof of Theorem 2.37(iii) $\Rightarrow$ (i)). Choose a nonprincipal ultrafilter $\mathcal{U}$ on I. Define the unital positive linear contractive mapping $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ by the formula

$$\forall F \in \mathrm{L}^\infty(G \times X), \quad \Phi(F) = \lim_{i\to\mathcal{U}} (\mu_i \otimes \mathrm{id}_X)(F).$$

The above limit is taken with respect to the weak*-topology in $\mathrm{L}^\infty(X)$.

(i) For every $f \in \mathrm{L}^\infty(X)$, we have

$$\Phi(\mathbf{1}_G \otimes f) = \lim_{i\to\mathcal{U}} (\mu_i \otimes \mathrm{id}_X)(\mathbf{1}_G \otimes f) = \lim_{i\to\mathcal{U}} \mu_i(\mathbf{1}_G)\, f = f.$$

(ii) For every $g \in G$ and every $F \in \mathrm{L}^\infty(G \times X)$, we have

$$\begin{aligned}
\Phi((\lambda \otimes \sigma)(g)F) &= \lim_{i\to\mathcal{U}} (\mu_i \otimes \mathrm{id}_X)((\lambda \otimes \sigma)(g))F) \\
&= \lim_{i\to\mathcal{U}} (\lambda(g^{-1})\mu_i \otimes \sigma(g))(F) \\
&= \lim_{i\to\mathcal{U}} (\mu_i \otimes \sigma(g))(F) \\
&= \sigma(g) \left(\lim_{i\to\mathcal{U}} (\mu_i \otimes \mathrm{id}_X)(F) \right) \\
&= \sigma(g)\Phi(F)
\end{aligned}$$

where in the third line we used the fact that $\|\lambda(g^{-1})\mu_i - \mu_i\|_1 \to 0$.

Thus, $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ is a G-equivariant projection and so the nonsingular action $G \curvearrowright (X, \nu)$ is amenable. $\square$

Next, we provide natural examples of amenable actions arising from homogeneous spaces.

THEOREM 2.44. *Let G be a locally compact second countable group and $H < G$ an amenable closed subgroup. Then the nonsingular left translation action $G \curvearrowright G/H$ is amenable.*

PROOF. Firstly, we show that the nonsingular left translation action $G \curvearrowright (G, m_G)$ is amenable. Fix $\mu \in \mathrm{P}(G)$. Define the unital positive linear contractive mapping $\Psi = \mu \otimes \mathrm{id}_G : \mathrm{L}^\infty(G \times G) \to \mathrm{L}^\infty(G)$. Then the following properties hold:

(i) For every $f \in \mathrm{L}^\infty(G)$, we have $\Psi(\mathbf{1}_G \otimes f) = \mu(\mathbf{1}_G) f = f$.
(ii) For every $g \in G$ and every $F \in \mathrm{L}^\infty(G \times G)$, we have

$$\Psi((\mathrm{id}_G \otimes \lambda)(g) F) = (\mu \otimes \lambda(g))(F) = \lambda(g)\Psi(F).$$

Next consider the nonsingular automorphism $\theta : G \times G \to G \times G : (h, k) \mapsto (kh, k)$ and define the unital positive linear contractive mapping $\Phi : \mathrm{L}^\infty(G \times G) \to \mathrm{L}^\infty(G)$ by the formula $\Phi(F) = \Psi(F \circ \theta)$ for every $F \in \mathrm{L}^\infty(G \times G)$. Then the following properties hold:

(i) For every $f \in \mathrm{L}^\infty(G)$, we have

$$\Phi(\mathbf{1}_G \otimes f) = \Psi((\mathbf{1}_G \otimes f) \circ \theta) = \Psi(\mathbf{1}_G \otimes f) = f.$$

(ii) For every $g \in G$ and every $F \in \mathrm{L}^\infty(G \times G)$, we have

$$\begin{aligned} \Phi((\lambda \otimes \lambda)(g) F) &= \Psi(F \circ (g^{-1} \otimes g^{-1}) \circ \theta) \\ &= \Psi(F \circ \theta \circ (\mathrm{id}_G \otimes g^{-1})) \\ &= \Psi((\mathrm{id}_G \otimes \lambda)(g)(F \circ \theta)) \\ &= \lambda(g)\Psi(F \circ \theta) \\ &= \lambda(g)\Phi(F). \end{aligned}$$

Thus, $\Phi : \mathrm{L}^\infty(G \times G) \to \mathrm{L}^\infty(G)$ is a G-equivariant projection and so the nonsingular translation action $G \curvearrowright (G, m_G)$ is amenable.

Secondly, let $H < G$ be an amenable closed subgroup. Consider the weak*-continuous right translation action $\rho : H \curvearrowright \mathrm{L}^\infty(G)$. Observe that we have the following identification of the fixed point subalgebra $\mathrm{L}^\infty(G)^{\rho(H)} = \mathrm{L}^\infty(G/H)$. Consider the unital weak*-continuous embedding

$$\iota : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(H \times G) : f \mapsto ((h, g) \mapsto f(gh)).$$

The embedding ι satisfies the following invariance property:

$$\forall h \in H, \forall f \in \mathrm{L}^\infty(G), \quad (\lambda(h) \otimes \rho(h))\iota(f) = \iota(f).$$

Since H is amenable, there exists a net of elements $(\mu_i)_{i \in I}$ in $\mathrm{P}(H)$ such that $\|\lambda(h)\mu_i - \mu_i\|_1 \to 0$ uniformly on compact subsets $K \subset H$ (see the proof of Theorem 2.37(iii) $\Rightarrow$ (i)). Choose a nonprincipal ultrafilter $\mathcal{U}$ on I.

Define the unital positive linear contractive mapping $\mathrm{E} : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)$ by the formula

$$\forall f \in \mathrm{L}^\infty(G), \quad \mathrm{E}(f) = \lim_{i \to \mathcal{U}} (\mu_i \otimes \mathrm{id}_G)(\iota(f)).$$

The above limit is taken with respect to the weak*-topology in $\mathrm{L}^\infty(G)$. For every $g \in G$ and every $f \in \mathrm{L}^\infty(G)$, we have $\mathrm{E}(\lambda(g) f) = \lambda(g)\,\mathrm{E}(f)$. For every $f \in \mathrm{L}^\infty(G)^{\rho(H)} = \mathrm{L}^\infty(G/H)$, we have $\iota(f) = \mathbf{1}_H \otimes f$ and so $\mathrm{E}(f) = f$. Moreover, for every $h \in H$ and every $f \in \mathrm{L}^\infty(G)$, we have

$$\begin{aligned}
\rho(h)\,\mathrm{E}(f) &= \rho(h)\left(\lim_{i \to \mathcal{U}} (\mu_i \otimes \mathrm{id}_G)(\iota(f))\right) \\
&= \lim_{i \to \mathcal{U}} (\mu_i \otimes \rho(h))(\iota(f)) \\
&= \lim_{i \to \mathcal{U}} (\lambda(h^{-1})\mu_i \otimes \rho(h))(\iota(f)) \\
&= \lim_{i \to \mathcal{U}} (\mu_i \otimes \mathrm{id}_G)\left((\lambda(h) \otimes \rho(h))\iota(f)\right) \\
&= \lim_{i \to \mathcal{U}} (\mu_i \otimes \mathrm{id}_G)(\iota(f)) = \mathrm{E}(f).
\end{aligned}$$

where in the third line we used the fact that $\|\lambda(h^{-1})\mu_i - \mu_i\|_1 \to 0$. This implies that $\mathrm{E} : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)$ is a $\lambda(G)$-equivariant unital positive linear contractive mapping such that $\mathrm{E}(\mathrm{L}^\infty(G)) = \mathrm{L}^\infty(G)^{\rho(H)} = \mathrm{L}^\infty(G/H)$ and $\mathrm{E}|_{\mathrm{L}^\infty(G/H)} = \mathrm{id}_{\mathrm{L}^\infty(G/H)}$.

Finally, regard $\mathrm{L}^\infty(G \times G/H) = \mathrm{L}^\infty(G \times G)^{(\mathrm{id}_G \otimes \rho)(H)} \subset \mathrm{L}^\infty(G \times G)$ and define $\Theta : \mathrm{E} \circ \Phi|_{\mathrm{L}^\infty(G \times G/H)} : \mathrm{L}^\infty(G \times G/H) \to \mathrm{L}^\infty(G/H)$. Then Θ is a G-equivariant projection and so the nonsingular left translation action $G \curvearrowright G/H$ is amenable. □

Finally, we observe that amenable actions are stable under taking restrictions to lattices.

Proposition 2.45. *Let G be a locally compact second countable group and $\Gamma < G$ a lattice. Then for any amenable nonsingular action $G \curvearrowright (X, \nu)$, the restriction $\Gamma \curvearrowright (X, \nu)$ is amenable.*

Proof. Denote by $\Phi : \mathrm{L}^\infty(G \times X) \to \mathrm{L}^\infty(X)$ the G-equivariant projection witnessing amenability of the nonsingular action $G \curvearrowright (X, \nu)$. Choose a Borel fundamental domain $\mathscr{F} \subset G$ so that $G = \mathscr{F} \cdot \Gamma$. Then $\mathscr{F}^{-1} \subset G$ is a Borel fundamental domain for the left translation action $\Gamma \curvearrowright G$. We may assume that $m_G(\mathscr{F}^{-1}) = 1$ so that $\eta = m_G|_{\mathscr{F}^{-1}} \in \mathrm{Prob}(\mathscr{F}^{-1})$. Then $\theta : (\Gamma \times \mathscr{F}^{-1}, m_\Gamma \otimes \eta) \to (G, m_G) : (\gamma, y) \mapsto \gamma y$ is a measure space isomorphism. Moreover, for all $\gamma, s \in \Gamma$ and all $y \in \mathscr{F}$, we have $\theta(\gamma s, y) = \gamma\theta(s, y)$. This implies that the canonical inclusion $\mathrm{L}^\infty(\Gamma \times X) \subset \mathrm{L}^\infty(\Gamma \times \mathscr{F}^{-1} \times X) \cong \mathrm{L}^\infty(G \times X)$ is Γ-equivariant. Thus $\Psi = \Phi|_{\mathrm{L}^\infty(\Gamma \times X)} : \mathrm{L}^\infty(\Gamma \times X) \to \mathrm{L}^\infty(X)$ is a Γ-equivariant projection. This shows that the nonsingular action $\Gamma \curvearrowright (X, \nu)$ is amenable. □

CHAPTER 3

Algebraic groups

We give an introduction to algebraic groups and their algebraic actions on algebraic varieties. We investigate the structure of stabilizers and the notion of tameness for algebraic actions. Standard references on linear algebraic groups are [**Bo91, Hu75**].

1. Algebraic varieties

We assume that K is an algebraically closed field of characteristic zero and that $k \subset K$ is a subfield. For every $n \geq 1$, we denote by $K[X_1, \ldots, X_n]$ (resp. $k[X_1, \ldots, X_n]$) the ring of polynomials in n indeterminates with coefficients in K (resp. k).

We say that $\mathbf{V} \subset K^n$ is an *affine algebraic variety* if there exists a subset $S \subset K[X_1, \ldots, X_n]$ such that

$$\mathbf{V} = \{(x_1, \ldots, x_n) \in K^n \mid \forall P \in S,\ P(x_1, \ldots, x_n) = 0\}\,.$$

We then denote by

$$I(\mathbf{V}) = \{P \in K[X_1, \ldots, X_n] \mid \forall (x_1, \ldots, x_n) \in \mathbf{V},\ P(x_1, \ldots, x_n) = 0\}$$

the *vanishing ideal* of $\mathbf{V}$ in $K[X_1, \ldots, X_n]$. We also denote by

$$K[\mathbf{V}] = K[X_1, \ldots, X_n]/I(\mathbf{V})$$

the ring of *regular functions* on $\mathbf{V}$. Hilbert's basis theorem shows that $K[\mathbf{V}]$ is a Noetherian ring. In particular, any ideal in $K[\mathbf{V}]$ is finitely generated. By Hilbert's Nullstellensatz, there is a one-to-one correspondence between affine algebraic varieties $\mathbf{V} \subset K^n$ and radical ideals $I \subset K[X_1, \ldots, X_n]$. Any intersection of affine algebraic varieties is again an algebraic variety and any finite union of algebraic varieties is again an algebraic variety. We define the *Zariski topology* on K^n by declaring that an algebraic variety $\mathbf{V} \subset K^n$ is a Zariski closed subset of K^n.

Let F be a vector space over K. A *k-structure* on F is a k-submodule $F_k \subset F$ such that the natural K-map $F_k \otimes_k K \to F$ is an isomorphism. A subspace $E \subset F$ is said to be *defined over* k or is a *k-subspace* if $E_k = E \cap F_k$ is k-structure on E, that is, $E = E_k \otimes_k K$. We have that $k[X_1, \ldots, X_n]$ is a k-structure on $K[X_1, \ldots, X_n]$. We say that $\mathbf{V} \subset K^n$ is an affine algebraic variety *defined over* k or is an *affine algebraic k-variety* if $I_k(\mathbf{V}) = I(\mathbf{V}) \cap k[X_1, \ldots, X_n]$ is a k-structure on $I(\mathbf{V})$. In that case, we denote by $\mathbf{V}(k) = \mathbf{V} \cap k^n$ the set of *k-points* of $\mathbf{V}$ and by $k[\mathbf{V}] = k[X_1, \ldots, X_n]/I_k(\mathbf{V})$ the ring of k-regular functions on $\mathbf{V}$. We regard $k[\mathbf{V}] \subset K[\mathbf{V}]$ via the well-defined injective mapping $k[\mathbf{V}] \to K[\mathbf{V}] : P+I_k(\mathbf{V}) \mapsto P+I(\mathbf{V})$. We naturally have $I(\mathbf{V}) = I_k(\mathbf{V}) \otimes_k K$ and $K[\mathbf{V}] = k[\mathbf{V}] \otimes_k K$. By definition, any algebraic variety $\mathbf{V}$ is defined over K. Note that $\mathbf{V} = \mathbf{V}(K)$ and $I_K(\mathbf{V}) = I(\mathbf{V})$.

We say that an affine algebraic variety $\mathbf{V} \subset K^n$ is *irreducible* if it cannot be written as a union of two proper Zariski closed subsets. Then the vanishing ideal $I(\mathbf{V})$ is prime and the ring of regular functions $K[\mathbf{V}]$ is an integral domain. We then denote by $K(\mathbf{V})$ the field of *rational functions* on $\mathbf{V}$, which is the field of fractions of $K[\mathbf{V}]$. More generally, any affine algebraic variety $\mathbf{V} \subset K^n$ can be written as a finite union of irreducible Zariski closed subsets. This follows from the fact that the ring $K[\mathbf{V}]$ is Noetherian.

Let $\mathbf{V} \subset K^n$ and $\mathbf{W} \subset K^p$ be affine algebraic varieties. We identify $K^n \times K^p$ with K^{n+p} and we endow K^{n+p} with the Zariski topolopy. Then the product $\mathbf{V} \times \mathbf{W} \subset K^{n+p}$ is an affine algebraic variety. If $\mathbf{V}$ and $\mathbf{W}$ are irreducible, then $\mathbf{V} \times \mathbf{W}$ is irreducible. In particular, K^n is irreducible for every $n \geq 1$.

Let $\mathbf{V} \subset K^n$ be an irreducible affine algebraic variety. The *dimension* $\dim(\mathbf{V})$ is defined as the transcendence degree of the field extension $K \subset K(\mathbf{V})$. For every $P \in I(\mathbf{V})$ and every $v = (v_1, \dots, v_n) \in \mathbf{V}$, define the differential $\mathrm{d}_v P = \sum_{i=1}^n \frac{\partial P}{\partial X_i}(v) X_i$. The *tangent space* $\mathscr{T}_v(\mathbf{V})$ at the point $v \in \mathbf{V}$ is defined as

$$\mathscr{T}_v(\mathbf{V}) = \{(x_1, \dots, x_n) \in K^n \mid \forall P \in I(\mathbf{V}),\ \mathrm{d}_v P(x_1, \dots, x_n) = 0\}.$$

Observe that if $\mathbf{V}$ is defined over k, then for every $v \in \mathbf{V}(k)$, $\mathscr{T}_v(\mathbf{V})$ has a natural k-structure $\mathscr{T}_v(\mathbf{V})_k \subset \mathscr{T}_v(\mathbf{V})$ and we have $\mathscr{T}_v(\mathbf{V}) = \mathscr{T}_v(\mathbf{V})_k \otimes_k K$. We always have $\dim_K(\mathscr{T}_v(\mathbf{V})) \geq \dim(\mathbf{V})$. We say that $v \in \mathbf{V}$ is a *simple point* if $\dim_K(\mathscr{T}_v(\mathbf{V})) = \dim(\mathbf{V})$. The set of simple points of $\mathbf{V}$ is a nonempty Zariski open set. We say that $\mathbf{V}$ is *smooth* if every point $v \in \mathbf{V}$ is simple. In particular, K^n is a smooth variety and $\dim(K^n) = n$ for every $n \geq 1$.

Let $\mathbf{V} \subset K^n$ be an affine algebraic variety. We say that $\mathbf{U} \subset \mathbf{V}$ is a *principal open set* if there exists a polynomial $P \in K[X_1, \dots, X_n]$ such that

$$\mathbf{U} = \{(x_1, \dots, x_n) \in \mathbf{V} \mid P(x_1, \dots, x_n) \neq 0\}.$$

Observe that $\mathbf{U}$ can be identified with the affine algebraic variety $\mathbf{W} \subset \mathbf{V} \times K \subset K^{n+1}$ defined by

$$\mathbf{W} = \{(x_1, \dots, x_n, t) \in \mathbf{V} \times K \mid P(x_1, \dots, x_n)t = 1\}.$$

Then we have $K[\mathbf{U}] = K[\mathbf{V}]([1/P])$. Any open set of $\mathbf{V}$ can be written as a union of principal open sets.

Let $\mathbf{V} \subset K^n$ and $\mathbf{W} \subset K^p$ be affine algebraic varieties. We say that $f : \mathbf{V} \to \mathbf{W}$ is a *regular map* or a *morphism* if for every $P \in K[\mathbf{W}]$, we have $P \circ f \in K[\mathbf{V}]$. For every $j \in \{1, \dots, p\}$, choose $P_j \in K[X_1, \dots, X_n]$ such that $f = (P_1 + I(\mathbf{V}), \dots, P_p + I(\mathbf{V}))$. For every $v \in \mathbf{V}$, we may define the *differential* $\mathrm{d}_v f : \mathscr{T}_v(\mathbf{V}) \to \mathscr{T}_{f(v)}(\mathbf{W})$ by the formula

$$\forall x = (x_1, \ldots, x_n) \in \mathscr{T}_v(\mathbf{V}), \quad (\mathrm{d}_v f)(x) = \left(\sum_{i=1}^{n} \frac{\partial P_j}{\partial X_i}(v) x_i \right)_j .$$

In case $\mathbf{V}$ and $\mathbf{W}$ are irreducible, we say that $f : \mathbf{V} \to \mathbf{W}$ is *dominant* if $f(\mathbf{V})$ is Zariski dense in $\mathbf{W}$. This amounts to saying that the map $f^* : K[\mathbf{W}] \to K[\mathbf{V}] : P \mapsto P \circ f$ is injective. We say that $f : \mathbf{V} \to \mathbf{W}$ is an *isomorphism* if f is a bijection and if both f and f^{-1} are regular maps. Assume moreover that $\mathbf{V} \subset K^n$ and $\mathbf{W} \subset K^p$ are affine algebraic k-varieties. Then we say that $f : \mathbf{V} \to \mathbf{W}$ is a *k-regular map* or a *k-morphism* if for every $P \in k[\mathbf{W}]$, we have $P \circ f \in k[\mathbf{V}]$. In that case, for every $v \in \mathbf{V}(k)$, we have $f(v) \in \mathbf{W}(k)$ and $(\mathrm{d}_v f)(\mathscr{T}_v(\mathbf{V})_k) \subset \mathscr{T}_{f(v)}(\mathbf{W})_k$. We say that $f : \mathbf{V} \to \mathbf{W}$ is a *k-isomorphism* if f is a bijection and if both f and f^{-1} are k-morphisms.

Consider the Galois group of the field extension $k \subset K$

$$\mathrm{Gal}(K/k) = \{\sigma \in \mathrm{Aut}(K) \mid \forall a \in k,\, \sigma(a) = a\} .$$

Then $\mathrm{Gal}(K/k)$ naturally acts on the polynomial ring $K[X_1, \ldots, X_n]$ in the following way: for every $P = \sum a_{i_1,\ldots,i_n} X_1^{i_1} \cdots X_n^{i_n} \in K[X_1, \ldots, X_n]$ and every $\sigma \in \mathrm{Gal}(K/k)$, define $P^\sigma = \sum \sigma(a_{i_1,\ldots,i_n}) X_1^{i_1} \cdots X_n^{i_n} \in K[X_1, \ldots, X_n]$. Let $\mathbf{V} \subset K^n$ be an affine algebraic variety and $\sigma \in \mathrm{Gal}(K/k)$. We may define the affine algebraic variety $\mathbf{V}^\sigma \subset K^n$ by the formula

$$\mathbf{V}^\sigma = \sigma(\mathbf{V}) = \{x \in K^n \mid \forall P \in I(\mathbf{V}),\ P^\sigma(x) = 0\} .$$

Then we have $I(\mathbf{V}^\sigma) = I(\mathbf{V})^\sigma$ and $K[\mathbf{V}^\sigma] = K[X_1, \ldots, X_n]/I(\mathbf{V})^\sigma$. Moreover, for every morphism of affine algebraic varieties $f : \mathbf{V} \to \mathbf{W}$ and every $\sigma \in \mathrm{Gal}(K/k)$, we may define the morphism $f^\sigma = \sigma f \sigma^{-1} : \mathbf{V}^\sigma \to \mathbf{W}^\sigma$.

It is useful to extend the notion of variety to the projective setting. Consider the projective space $\mathrm{P}^n = \mathrm{P}^n(K) = \mathrm{P}(K^{n+1})$. We say that $\mathbf{V} \subset \mathrm{P}^n$ is a *projective algebraic variety* if there exists a subset $S \subset K[X_0, \ldots, X_n]$ consisting of homogeneous polynomials such that

$$\mathbf{V} = \{(x_0, \ldots, x_n) \in \mathrm{P}^n \mid \forall P \in S,\ P(x_0, \ldots, x_n) = 0\} .$$

We say that $\mathbf{V} \subset \mathrm{P}^n$ is a *projective algebraic k-variety* if there exists a subset $S \subset k[X_0, \ldots, X_n]$ consisting of homogeneous polynomials such that

$$\mathbf{V} = \{(x_0, \ldots, x_n) \in \mathrm{P}^n \mid \forall P \in S,\ P(x_0, \ldots, x_n) = 0\} .$$

We then denote by $\mathbf{V}(k) = \mathbf{V} \cap \mathrm{P}(k^{n+1})$ the set of *k-points* of $\mathbf{V}$. As in the affine case, we can define the Zariski topology on P^n. A *quasiprojective algebraic k-variety* is a Zariski open set defined over k in a projective algebraic k-variety. Observe that any affine algebraic k-variety can be regarded as a quasiprojective algebraic k-variety. More generally, one can define the notion of abstract *algebraic k-variety* that generalizes the notion of quasiprojective algebraic k-variety. All examples of algebraic k-varieties we consider in these notes are quasiprojective algebraic k-varieties.

The following result provides a criterion for an algebraic variety to be defined over k and for a morphism between algebraic varieties to be defined over k. For further details, we refer the reader to [**Bo91**, AG 14].

PROPOSITION 3.1. *The following assertions hold:*

(i) *Let* $\mathbf{V}$ *be an algebraic variety. Then* $\mathbf{V}$ *is defined over* k *if and only if* $\mathbf{V}^\sigma = \mathbf{V}$ *for every* $\sigma \in \mathrm{Gal}(K/k)$.

(ii) *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be algebraic* k*-varieties and* $f : \mathbf{V} \to \mathbf{W}$ *a morphism. Then* f *is defined over* k *if and only if* $f^\sigma = f$ *for every* $\sigma \in \mathrm{Gal}(K/k)$.

PROOF. (i) We may and will assume that $\mathbf{V} \subset K^n$ is an affine algebraic variety. If $\mathbf{V}$ is defined over k, then $I(\mathbf{V}) = I_k(\mathbf{V}) \otimes_k K$. Then for every $\sigma \in \mathrm{Gal}(K/k)$, we have $I(\mathbf{V})^\sigma = I(\mathbf{V})$ and so $\mathbf{V}^\sigma = \mathbf{V}$. Conversely, assume that $\mathbf{V}^\sigma = \mathbf{V}$ for every $\sigma \in \mathrm{Gal}(K/k)$. It suffices to show that $I(\mathbf{V})$ is generated by $I_k(\mathbf{V})$. Denote by $J \subset I(\mathbf{V})$ the ideal generated by $I_k(\mathbf{V})$. Then $F = K[X_1, \dots, X_n]/J$ has a k-structure $F_k = k[X_1, \dots, X_n]/I_k(\mathbf{V})$ so that $F = F_k \otimes_k K$. By contradiction, assume that $J \neq I(\mathbf{V})$. Set $E = I(\mathbf{V})/J \subset K[X_1, \dots, X_n]/J$ and note that $E_k = E \cap F_k = \{0\}$. Choose a basis $(e_i)_i$ of F_k and choose $w \in E\backslash\{0\}$ such that w can be expressed with a minimal number of elements of $(e_i)_i$. Upon multiplying w by a scalar in K^*, we may assume that $w = e_{i_1} + \alpha_2 e_{i_2} + \cdots + \alpha_r e_{i_r}$ with $r \geq 2$, $\alpha_2, \dots, \alpha_r \in K$ and $\alpha_2 \notin k$. Then there exists $\sigma \in \mathrm{Gal}(K/k)$ such that $\sigma(\alpha_2) \neq \alpha_2$. Then $w - \sigma(w) = (\alpha_2 - \sigma(\alpha_2))e_{i_2} + \cdots + (\alpha_r - \sigma(\alpha_r))e_{i_r}$ and $w - \sigma(w) \in E$ because $I(\mathbf{V})^\sigma = I(\mathbf{V})$. Since $w - \sigma(w) \neq 0$, we obtain a contradiction on the minimal number of elements of $(e_i)_i$. Therefore $I(\mathbf{V}) = J$ is generated by $I_k(\mathbf{V})$ and so $\mathbf{V}$ is defined over k.

(ii) We may and will assume that $\mathbf{V} \subset K^n$ and $\mathbf{W} \subset K^p$ are affine algebraic k-varieties. We have $K[\mathbf{V}] = k[\mathbf{V}] \otimes_k K$ and $K[\mathbf{W}] = k[\mathbf{W}] \otimes_k K$. Denote by $f^* : K[\mathbf{W}] \to K[\mathbf{V}] : P \mapsto P \circ f$ the associated K-algebra homomorphism. If f is defined over k, then $f^*(k[\mathbf{W}]) \subset k[\mathbf{V}]$. It follows that $f^\sigma = f$ for every $\sigma \in \mathrm{Gal}(K/k)$. Conversely, assume that $f^\sigma = f$ for every $\sigma \in \mathrm{Gal}(K/k)$. Let $P \in k[\mathbf{W}]$. For every $\sigma \in \mathrm{Gal}(K/k)$, we have $(f^*(P))^\sigma = (P \circ f)^\sigma = P^\sigma \circ f^\sigma = P \circ f = f^*(P)$ and so $f^*(P) \in k[\mathbf{V}]$. Therefore we have $f^*(k[\mathbf{W}]) \subset k[\mathbf{V}]$ and so f is defined over k. □

The following useful result provides another sufficient condition for an algebraic variety to be defined over k and for a morphism between algebraic varieties to be defined over k.

PROPOSITION 3.2. *The following assertions hold:*

(i) *Let* $\mathbf{V}$ *be an algebraic* k*-variety and* $B \subset \mathbf{V}(k)$ *a nonempty subset. Denote by* $\mathbf{W}$ *the Zariski closure of* B *in* $\mathbf{V}$. *Then* $\mathbf{W}$ *is defined over* k.

(ii) *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be algebraic* k*-varieties and* $f : \mathbf{V} \to \mathbf{W}$ *a morphism. Let* $B \subset \mathbf{V}(k)$ *be a Zariski dense subset such that* $f(B) \subset \mathbf{W}(k)$. *Then* f *is defined over* k.

PROOF. (i) We may and will assume that $\mathbf{V} \subset K^n$ is an affine algebraic k-variety. Then $\mathbf{V}(k) \subset k^n$. Denote by $I(\mathbf{W}) \subset K[X_1, \dots, X_n]$ the vanishing ideal of $\mathbf{W}$. For every $d \geq 1$, set $I(\mathbf{W})_d = \{P \in I(\mathbf{W}) \mid \deg(P) \leq d\}$. We then have $I(\mathbf{W}) = \bigcup_{d=1}^{\infty} I(\mathbf{W})_d$. Denote by r the number of n-tuples $(i_1, \dots, i_n) \in \mathbb{N}^n$ such that $\sum_{j=1}^{n} i_j \leq d$. Any polynomial $P \in I(\mathbf{W})_d$ has at most r coefficients in K and so we may identify P with a r-tuple $(\alpha_1, \dots, \alpha_r) \in K^r$. Since $B \subset \mathbf{V}(k)$ is Zariski dense in $\mathbf{W}$, we have $P \in I(\mathbf{W})_d$ if and only if $P(g) = 0$ for every $g \in B$. We can regard the system of equations $P(g) = 0$ for $g \in B$ as a system of linear equations with coefficients in k and variables $\alpha_1, \dots, \alpha_r$. Then there are at most r linear equations such that the solutions of the original system of equations are exactly the same as the solutions of these r linear equations. In other words, there is a linear transformation $T : K^r \to K^r$ for which $P \in I(\mathbf{W})_d$ if and only if $(\alpha_1, \dots, \alpha_r) \in \ker(T)$. Since the matrix representation of T with respect to the canonical basis of K^r lies in $\mathrm{M}_r(k)$, it follows that $\ker(T)$ has a basis that we can choose in k^r. This implies that $I(\mathbf{W})_d$ is generated by $I(\mathbf{W})_d \cap k[X_1, \dots, X_n]$. Since this is true for every $d \geq 1$, this implies that $I(\mathbf{W})$ is generated by $I(\mathbf{W}) \cap k[X_1, \dots, X_n]$ and so $\mathbf{W}$ is defined over k.

(ii) We may and will assume that $\mathbf{V} \subset K^n$ and $\mathbf{W} \subset K^p$ are algebraic affine k-varieties. Moreover using coordinate functions, we may further assume that $\mathbf{W} = K$. Then we have $f \in K[\mathbf{V}] = K[X_1, \dots, X_n]/I(\mathbf{V})$. Regarding $f : \mathbf{V} \to K$, we have $f(B) \subset k$. Choose $P \in K[X_1, \dots, X_n]$ such that $P + I(\mathbf{V}) = f$. Write $P = P_0 + \sum_{i=1}^{r} \alpha_i P_i$ where $P_0, P_1, \dots, P_r \in k[X_1, \dots, X_n]$, $\alpha_1, \dots, \alpha_r \in K$ and $1, \alpha_1, \dots, \alpha_r$ are linearly independent over k. Since $f(B) \subset k$, we have $P(B) \subset k$ and the linear independence of $1, \alpha_1, \dots, \alpha_r$ over k implies that $P(v) = P_0(v)$ for every $v \in B$. Since $B \subset \mathbf{V}(k)$ is Zariski dense, it follows that $P = P_0 \in k[X_1, \dots, X_n]$. This implies that $f = P_0 + I(\mathbf{V}) = P_0 + I_k(\mathbf{V})$ is defined over k. $\square$

We will need the following classical result regarding morphisms between algebraic varieties.

THEOREM 3.3 (Chevalley). *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be algebraic varieties and* $f : \mathbf{V} \to \mathbf{W}$ *a morphism. Then* $\overline{f(\mathbf{V})}$ *contains a Zariski open dense subset* $\mathbf{U}$ *such that* $\mathbf{U} \subset f(\mathbf{V})$.

PROOF. We may and will assume that $\mathbf{V}$ and $\mathbf{W}$ are affine algebraic varieties. We may further assume that $\mathbf{V}$ and $\mathbf{W}$ are irreducible. Indeed, denote by $\mathbf{V}_1, \dots, \mathbf{V}_k$ the irreducible components of $\mathbf{V}$ so that $\mathbf{V} = \mathbf{V}_1 \cup \dots \cup \mathbf{V}_k$. For every $i \in \{1, \dots, k\}$, set $\mathbf{W}_i = \overline{f(\mathbf{V}_i)}$. Since $\mathbf{W}_1 \cup \dots \cup \mathbf{W}_k$ is closed and contains $f(\mathbf{V})$, it follows that $\mathbf{W}_1 \cup \dots \cup \mathbf{W}_k = \overline{f(\mathbf{V})}$. Moreover, for every $i \in \{1, \dots, k\}$, $\mathbf{W}_i$ is irreducible. Therefore, without loss of generality, we may and will assume that $\mathbf{V}$ and $\mathbf{W}$ are irreducible affine algebraic varieties.

Set $A = K[\mathbf{W}]$ and $B = K[\mathbf{V}]$. Then the map $f^* : A \to B : P \mapsto P \circ f$ is injective. We prove the following technical result from commutative algebra.

CLAIM 3.4. For every $b \in B \setminus \{0\}$, there exists $a \in A$ such that for every K-algebra homomorphism $\varphi : A \to K$ such that $\varphi(a) \neq 0$, there exists a K-algebra homomorphism $\overline{\varphi} : B \to K$ such that $\varphi = \overline{\varphi} \circ f^*$ and $\overline{\varphi}(b) \neq 0$.

PROOF OF CLAIM 3.4. We may identify $A = f^*(A)$ and regard $A \subset B$. Since B is finitely generated over A, by induction over the number of generators, we may assume that there exists $x \in B$ such that $B = A[x]$. Let T be an indeterminate variable. Then B is a quotient of $A[T]$. Denote by L the fraction field of A. We regard $A \subset L$ and $A[T] \subset L[T]$. Let $b \in B \setminus \{0\}$. There are two cases to consider.

Firstly, assume that $B \cong A[T]$. Since $b = Q \in A[T] \setminus \{0\}$, there exists $c \in A$ such that $a = Q(c) \neq 0$. For every K-algebra homomorphism $\varphi : A \to K$, consider the canonical extension $\overline{\varphi} : A[T] \to K : P \mapsto \varphi(P(c))$. Then $\overline{\varphi}(b) = \overline{\varphi}(Q) = \varphi(Q(c)) = \varphi(a)$ and we are done.

Secondly, assume that B is a proper quotient of $A[T]$. Then we may choose a nonzero polynomial $P = \sum_{i=0}^{d} p_i T^i \in A[T]$ of minimal degree such that $P(x) = 0$. Then the principal ideal $I(x) = \{R \in L[T] \mid R(x) = 0\}$ is generated by P. We may regard $B \subset L[T]/I(x)$ so that we may write $b = Q_0(x)$ for some $Q_0 \in L[T]$ such that $\deg(Q_0) \leq d-1$. Upon multiplying Q_0 by an element of A, we obtain a polynomial $Q = \sum q_i T^i \in A[T]$ such that $\deg(Q) \leq d-1$ and such that b divides $Q(x)$.

Since $B = A[x]$ is an integral domain, it follows that the polynomial $P \in A[T]$ is irreducible in $L[T]$. Since the characteristic of L is zero, it follows that P has exactly d distinct roots and so the gcd of P and P' is equal to 1. This further implies that there exists $r \in A \setminus \{0\}$ and $R_1, R_2 \in A[T]$ such that $r = R_1 P + R_2 P'$. Set $a = r p_d q$ where $q \in A$ is a nonzero coefficient of $Q \in A[T]$.

Let now $\varphi : A \to K$ be a K-algebra homomorphism such that $\varphi(a) \neq 0$. Consider the canonical extension $\Phi : A[T] \to K[T] : \sum a_i T^i \mapsto \sum \varphi(a_i) T^i$. Since $\varphi(p_d) \neq 0$, the polynomial $\Phi(P)$ has degree d. Since $\Phi(R_1)\Phi(P) + \Phi(R_2)\Phi(P)' = \varphi(r) \neq 0$, the polynomial $\Phi(P)$ has exactly d roots in K. Since $\varphi(q) \neq 0$, the polynomial $\Phi(Q)$ is nonzero and has degree less than or equal to $d-1$. We may choose a root $\lambda \in K$ of $\Phi(P)$ that is not a root of $\Phi(Q)$. This implies that $\sum \varphi(q_i)\lambda^i \neq 0$. Consider the K-algebra homomorphism $\overline{\varphi} : A[T] \to K : \sum a_i T^i \mapsto \sum \varphi(a_i)\lambda^i$. Denote by $\mathscr{I} = \ker(A[T] \to B)$. We claim that $\overline{\varphi}(\mathscr{I}) = 0$. Indeed, let $R \in \mathscr{I}$. Then $R \in I(x)$ and so there exists $R_0 \in L[T]$ such that $R = R_0 P \in A[T]$. If $R = 0$, it is clear that $\overline{\varphi}(R) = 0$. If $R \neq 0$, set $n - 1 = \deg(R_0)$ with $n \geq 1$. Then it is easy to see that $p_d^n R_0 \in A[T]$. It follows that $p_d^n R = (p_d^n R_0)P$ and so $\overline{\varphi}(p_d^n R) = \overline{\varphi}(p_d^n R_0)\overline{\varphi}(P) = 0$. Since $\varphi(p_d^n) \neq 0$, it follows that $\overline{\varphi}(R) = 0$. Therefore, $\overline{\varphi} : B \to K : \sum a_i x^i \mapsto \sum \varphi(a_i)\lambda^i$ is a well-defined K-algebra homomorphism such that $\overline{\varphi}|_A = \varphi$ and such that $\overline{\varphi}(b) \neq 0$ since $\overline{\varphi}(b)$ divides $\sum \varphi(q_i)\lambda^i \neq 0$. This finishes the proof of the claim. □

We may apply Claim 3.4 to $b = 1$ to obtain a regular function $a \in A = K[\mathbf{W}]$ satisfying the conclusion of the claim. Consider the principal open set $\mathbf{U}_a = \{w \in \mathbf{W} \mid a(w) \neq 0\}$. For every $w \in \mathbf{U}_a$, consider the K-algebra homomorphism $\varphi_w : K[\mathbf{W}] \to K : P \mapsto P(w)$. Since $\varphi_w(a) = a(w) \neq 0$, there exists a K-algebra homomorphism $\overline{\varphi}_w : B \to K$ such that $\varphi_w = \overline{\varphi}_w \circ f^*$. Note that there exists a unique point $v \in \mathbf{V}$ such that $\overline{\varphi}_w = \varphi_v : K[\mathbf{V}] \to K : P \mapsto P(v)$. Then we have $\varphi_w = \overline{\varphi}_w \circ f^* = \varphi_v \circ f^* = \varphi_{f(v)}$ and so $w = f(v) \in f(\mathbf{V})$. This shows that $\mathbf{U}_a \subset f(\mathbf{V})$ and finishes the proof of the theorem. □

It is possible to further refine Chevalley's Theorem to obtain the following precise statement.

Corollary 3.5. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be irreducible algebraic varieties and* $f : \mathbf{V} \to \mathbf{W}$ *a dominant morphism. The following assertions hold:*

(i) *There exists a Zariski dense open set* $\mathbf{U} \subset \mathbf{V}$ *such that for every* $u \in \mathbf{U}$*, the differential* $\mathrm{d}_u f : \mathscr{T}_u(\mathbf{V}) \to \mathscr{T}_{f(u)}(\mathbf{W})$ *is surjective.*

(ii) *Assume that* $\mathbf{V}$ *and* $\mathbf{W}$ *are defined over* k *and that* $f : \mathbf{V} \to \mathbf{W}$ *a bijective* k*-morphism. Then there exists a Zariski dense open set* $\mathbf{W}_0 \subset \mathbf{W}$ *such that* $\mathbf{W}_0$ *is defined over* k *and such that* $f : f^{-1}(\mathbf{W}_0) \to \mathbf{W}_0$ *is a* k*-isomorphism.*

We refer the reader to [**Hu75**, Section I.4] for further details.

2. Algebraic groups

2.1. Generalities. For every $n \geq 1$, we identify the space $\mathrm{M}_n = \mathrm{M}_n(K)$ with K^{n^2} in the usual way. Recall that $\det \in K[X_{ij} \mid 1 \leq i, j \leq n]$. Then the linear group $\mathrm{GL}_n = \mathrm{GL}_n(K) = \det^{-1}(K \setminus \{0\})$ is a principal open set of K^{n^2}. It can also be regarded as the affine algebraic variety

$$\mathrm{GL}_n = \{(A, t) \in \mathrm{M}_n(K) \times K \mid \det(A)t = 1\} \subset K^{n^2+1}.$$

Definition 3.6. A *linear algebraic group* $\mathbf{G}$ is a Zariski closed subgroup $\mathbf{G} < \mathrm{GL}_n$. We say that $\mathbf{G}$ is a *linear algebraic* k*-group* if the affine algebraic variety $\mathbf{G} \subset \mathrm{GL}_n$ is defined over k. Then we define the group of its k-points by $\mathbf{G}(k) = \mathbf{G} \cap \mathrm{GL}_n(k)$.

By definition, GL_n is a linear algebraic group and we have

$$\begin{aligned} K[\mathrm{GL}_n] &= K[X_{ij}, Z \mid 1 \leq i, j \leq n]/(\det((X_{ij})_{ij})Z = 1) \\ &= K[X_{ij}, \det((X_{ij})_{ij})^{-1} \mid 1 \leq i, j \leq n]. \end{aligned}$$

All the algebraic groups we consider in these notes are assumed to be linear.

Examples 3.7. Here are some classical examples of (linear) algebraic groups. All the following algebraic groups are defined over $\mathbb{Q}$.

(1) The additive group $\mathbf{G}_a = (K, +)$ can be regarded as

$$\mathbf{G}_a = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in K \right\} < \mathrm{GL}_2\,.$$

Then we have $K[\mathbf{G}_a] = K[T]$.

(2) The multiplicative group $\mathbf{G}_m = (K^*, \times)$ can be regarded as

$$\mathbf{G}_m = \left\{ \begin{pmatrix} z & 0 \\ 0 & t \end{pmatrix} \mid z, t \in K,\, zt = 1 \right\} < \mathrm{GL}_2 .$$

Then we have $K[\mathbf{G}_m] = K[Z,T]/(ZT = 1) = K[T, T^{-1}]$.

(3) The special linear group SL_n is defined as

$$\mathrm{SL}_n = \{g \in \mathrm{GL}_n \mid \det(g) = 1\} < \mathrm{GL}_n .$$

Then $K[\mathrm{SL}_n] = K[X_{ij} \mid 1 \leq i, j \leq n]/(\det((X_{ij})_{ij}) = 1)$.

(4) The projective linear group PGL_n can be regarded as a linear algebraic group as follows. Consider the algebra $\mathrm{M}_n = \mathrm{M}_n(K)$ and its automorphism group $\mathrm{Aut}(\mathrm{M}_n)$. The map $\iota : \mathrm{PGL}_n \to \mathrm{Aut}(\mathrm{M}_n) : g \mapsto (X \mapsto gXg^{-1})$ is a well-defined injective group homomorphism. By Skolem–Noether theorem, $\iota : \mathrm{PGL}_n \to \mathrm{Aut}(\mathrm{M}_n)$ is onto and so we may identify PGL_n with $\mathrm{Aut}(\mathrm{M}_n)$. Using the usual identification of M_n with K^{n^2}, we may then regard $\mathrm{Aut}(\mathrm{M}_n) < \mathrm{GL}_{n^2}$ as a Zariski closed subgroup.

We record the following useful fact.

Lemma 3.8. *Let $H < \mathrm{GL}_n$ be a subgroup and denote by $\mathbf{H} = \overline{H} \subset \mathrm{GL}_n$ its Zariski closure. Then $\mathbf{H} < \mathrm{GL}_n$ is an algebraic group.*

Proof. It suffices to prove that $\mathbf{H} < \mathrm{GL}_n$ is a subgroup. Firstly, the inversion map $m : \mathrm{GL}_n \to \mathrm{GL}_n : x \mapsto x^{-1}$ is an isomorphism of algebraic varieties. Since $H^{-1} = H$, it follows that

$$\mathbf{H}^{-1} = \overline{H}^{-1} = \overline{H^{-1}} = \overline{H} = \mathbf{H}$$

and so $\mathbf{H}$ is stable under inversion. Secondly, for every $g \in \mathrm{GL}_n$, the left multiplication $L_g : \mathrm{GL}_n \to \mathrm{GL}_n : x \mapsto gx$ is an isomorphism of algebraic varieties. If $g \in H$, then $gH = H$ and so we have

$$g\mathbf{H} = g\overline{H} = \overline{gH} = \overline{H} = \mathbf{H}.$$

Likewise, for every $h \in \mathrm{GL}_n$, the right multiplication $R_h : \mathrm{GL}_n \to \mathrm{GL}_n : x \mapsto xh$ is an isomorphism of algebraic varieties. If $h \in \mathbf{H}$, then the above equality implies that $Hh \subset \mathbf{H}$ and so we have

$$\mathbf{H}h = \overline{H}h = \overline{Hh} \subset \overline{\mathbf{H}} = \mathbf{H}.$$

Thus, $\mathbf{H}$ is stable under multiplication and so $\mathbf{H} < \mathrm{GL}_n$ is a subgroup. □

Next, we turn to general structural properties of algebraic groups.

Proposition 3.9. *Let $\mathbf{G}$ be an algebraic group. The following assertions hold:*

(i) *The Zariski connected components of $\mathbf{G}$ coincide with the Zariski irreducible components of $\mathbf{G}$.*

(ii) *Denote by* $\mathbf{G}^0$ *the Zariski connected component of the identity element* $e \in \mathbf{G}$*. Then* $\mathbf{G}^0 \lhd \mathbf{G}$ *is a normal Zariski closed subgroup and has finite index. Moreover, any Zariski connected closed subgroup of* $\mathbf{G}$ *is contained in* $\mathbf{G}^0$*. If* $\mathbf{G}$ *is defined over* k*, then so is* $\mathbf{G}^0$*.*

(iii) *Any finite index Zariski closed subgroup of* $\mathbf{G}$ *contains* $\mathbf{G}^0$*.*

We say that an algebraic group $\mathbf{G}$ *is (Zariski) connected if* $\mathbf{G} = \mathbf{G}^0$*.*

Proof. (i) Denote by $\mathbf{Z}_1, \dots, \mathbf{Z}_n$ the irreducible components of $\mathbf{G}$ that contain the identity element $e \in \mathbf{G}$. Denote by $\mathbf{Z} = \mathbf{Z}_1 \cdots \mathbf{Z}_n \subset \mathbf{G}$ the range of the morphism $\mathbf{Z}_1 \times \cdots \times \mathbf{Z}_n \to \mathbf{G} : (z_1, \dots, z_n) \mapsto z_1 \cdots z_n$. Since $\mathbf{Z}_1 \times \cdots \times \mathbf{Z}_n$ is irreducible, $\mathbf{Z}$ is irreducible and contains $e \in \mathbf{G}$. Then there exists $i \in \{1, \dots, n\}$ such that $\mathbf{Z} \subset \mathbf{Z}_i$. Since $\mathbf{Z}_j \subset \mathbf{Z}$ for every $j \in \{1, \dots, n\}$, we infer that there exists a unique irreducible component that contains $e \in \mathbf{G}$. We denote this unique irreducible component by $\mathbf{G}^0$. Then $\mathbf{G}^0 \subset \mathbf{G}$ is Zariski closed and the previous reasoning shows that $\mathbf{G}^0$ is stable under multiplication.

For every $g \in \mathbf{G}^0$, $g^{-1}\mathbf{G}^0$ is the range of the morphism $\mathbf{G}^0 \to \mathbf{G} : h \mapsto g^{-1}h$ and so $g^{-1}\mathbf{G}^0$ is irreducible. Since $e \in g^{-1}\mathbf{G}^0$, we have $g^{-1}\mathbf{G}^0 \subset \mathbf{G}^0$. This further implies that $\mathbf{G}^0$ is stable under inverse and so $\mathbf{G}^0 < \mathbf{G}$ is an algebraic subgroup. For every $g \in \mathbf{G}$, $g\mathbf{G}^0 g^{-1}$ is the range of the morphism $\mathbf{G}^0 \to \mathbf{G} : h \mapsto ghg^{-1}$ and so $g\mathbf{G}^0 g^{-1}$ is irreducible. Since $e \in g\mathbf{G}^0 g^{-1}$, we have $g\mathbf{G}^0 g^{-1} \subset \mathbf{G}^0$. Likewise, we have $g^{-1}\mathbf{G}^0 g \subset \mathbf{G}^0$. This shows that $\mathbf{G}^0 \lhd \mathbf{G}$ is a normal subgroup.

For every $g \in \mathbf{G}^0$, the subset $g\mathbf{G}^0$ is an irreducible component of $\mathbf{G}$ hence connected. Since $K[\mathbf{G}]$ is Noetherian, $\mathbf{G}$ has only finitely many irreducible components and so $\mathbf{G}^0 < \mathbf{G}$ has finite index. This further implies that $g\mathbf{G}^0$ is Zariski open and closed for every $g \in \mathbf{G}$. Therefore, $(g\mathbf{G}^0)_{g \in \mathbf{G}}$ are the Zariski connected components of $\mathbf{G}$.

(ii) We already proved that $\mathbf{G}^0 \lhd \mathbf{G}$ is a normal Zariski closed subgroup and has finite index. Let $\mathbf{H} < \mathbf{G}$ be a Zariski connected closed subgroup. Since $e \in \mathbf{H}$ and since $\mathbf{G}^0$ is the Zariski connected component of $e \in \mathbf{G}$, it follows that $\mathbf{H} < \mathbf{G}^0$. Assume further that $\mathbf{G}$ is defined over k. For every $\sigma \in \mathrm{Gal}(K/k)$, $(\mathbf{G}^0)^\sigma$ is the Zariski connected component in $\mathbf{G}^\sigma = \mathbf{G}$ of the identity element $e \in \mathbf{G}$ and so $(\mathbf{G}^0)^\sigma = \mathbf{G}^0$. Thus, $\mathbf{G}^0 < \mathbf{G}$ is a k-subgroup.

(iii) Let $\mathbf{H} < \mathbf{G}$ be a finite index Zariski closed subgroup. Then $\mathbf{H} < \mathbf{G}$ is also Zariski open and so $\mathbf{H}$ is a union of Zariski connected components of $\mathbf{G}$. Since $e \in \mathbf{H}$, we have $\mathbf{G}^0 < \mathbf{H}$. □

All the examples considered in Examples 3.7 are connected algebraic groups.

A *k-homomorphism* $\varphi : \mathbf{G} \to \mathbf{H}$ of algebraic k-groups is a k-morphism of algebraic k-varieties that is also a group homomorphism. Note that $\ker(\varphi) \lhd \mathbf{G}$ is a normal algebraic subgroup. Moreover, using Proposition 3.1, for every $\sigma \in \mathrm{Gal}(K/k)$ and every $h \in \ker(\varphi)$, we have $\varphi(\sigma(h)) = \sigma(\varphi(h)) = 0$ and so $\ker(\varphi)^\sigma = \ker(\varphi)$. Thus, $\ker(\varphi) \lhd \mathbf{G}$ is a k-subgroup.

A *k-representation* of an algebraic k-group $\mathbf{G}$ in a finite dimensional k-vector space $\mathbf{V}$ is a k-homomorphism $\pi : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$.

DEFINITION 3.10. Let $\mathbf{G}$ be a connected algebraic k-group. We say that $\mathbf{G}$ is

- *semisimple* if the only abelian normal connected algebraic subgroup of $\mathbf{G}$ is $\{e\}$.
- *simple* if $\mathbf{G}$ is not abelian and if the only proper normal algebraic subgroup of $\mathbf{G}$ is $\{e\}$.
- *almost simple* if $\mathbf{G}$ is not abelian and if the only proper normal algebraic subgroups of $\mathbf{G}$ are finite.

EXAMPLES 3.11. For every $n \geq 2$, we have that

- SL_n is an almost simple connected algebraic group.
- PGL_n is a simple connected algebraic group.

2.2. The Lie algebra of an algebraic group. Since GL_n is a Zariski open set of M_n, its tangent space $\mathscr{T}_e(\mathrm{GL}_n)$ at $e \in \mathrm{GL}_n$ is the k-vector space $\mathrm{M}_n = \mathrm{End}(K^n)$. More generally, let $\mathbf{G} < \mathrm{GL}_n$ be an algebraic k-group. Then its tangent space $\mathscr{T}_e(\mathbf{G})$ at $e \in \mathbf{G}$ is naturally a k-subspace of M_n that we denote by $\mathrm{Lie}(\mathbf{G})$. It is the *Lie algebra* of $\mathbf{G}$. By definition, we have $\mathrm{Lie}(\mathrm{GL}_n) = \mathrm{M}_n$.

For every $g \in \mathbf{G}$, consider the inner automorphism $\mathrm{inn}(g) : \mathbf{G} \to \mathbf{G} : x \mapsto gxg^{-1}$. We denote by $\mathrm{Ad}(g) = \mathrm{d}_e(\mathrm{inn}(g)) : \mathrm{Lie}(\mathbf{G}) \to \mathrm{Lie}(\mathbf{G}) : X \mapsto gXg^{-1}$ its differential at $e \in \mathbf{G}$. Then the map $\mathrm{Ad} : \mathbf{G} \to \mathrm{GL}(\mathrm{Lie}(\mathbf{G})) : g \mapsto \mathrm{Ad}(g)$ is a k-representation called the *adjoint representation* of $\mathbf{G}$.

The differential $\mathrm{ad} = \mathrm{d}_e(\mathrm{Ad}) : \mathrm{Lie}(\mathbf{G}) \to \mathrm{End}(\mathrm{Lie}(\mathbf{G})) : X \mapsto (Y \mapsto XY - YX)$ at $e \in \mathbf{G}$ is called the *adjoint representation* of $\mathrm{Lie}(\mathbf{G})$. We then simply denote by $[\,\cdot\,,\cdot\,] : \mathrm{Lie}(\mathbf{G}) \times \mathrm{Lie}(\mathbf{G}) \to \mathrm{Lie}(\mathbf{G}) : (X, Y) \mapsto \mathrm{ad}(X)(Y) = XY - YX$ the Lie bracket on $\mathrm{Lie}(\mathbf{G})$. If $\mathbf{H} < \mathbf{G}$ is a k-subgroup, then $\mathrm{Lie}(\mathbf{H}) \subset \mathrm{Lie}(\mathbf{G})$ is a Lie k-subalgebra.

2.3. Algebraic actions of algebraic groups.

DEFINITION 3.12. Let $\mathbf{G}$ be an algebraic k-group and $\mathbf{V}$ an algebraic k-variety. An *algebraic k-action* $\mathbf{G} \curvearrowright \mathbf{V}$ is an action for which the map $\mathbf{G} \times \mathbf{V} \to \mathbf{V} : (g, v) \mapsto gv$ is a k-morphism. We simply say that $\mathbf{V}$ is an *algebraic k-$\mathbf{G}$-variety.*

We say that $\mathbf{V}$ is a *homogeneous* algebraic k-$\mathbf{G}$-variety if the action $\mathbf{G} \curvearrowright \mathbf{V}$ is transitive. The next result shows that orbits of algebraic actions of algebraic groups are well-behaved.

PROPOSITION 3.13. *Let $\mathbf{G}$ be an algebraic k-group and $\mathbf{V}$ an algebraic k-$\mathbf{G}$-variety. Then for every $v \in \mathbf{V}$, the orbit $\mathbf{G}v$ is locally closed in $\mathbf{V}$ for the Zariski topology. If $v \in \mathbf{V}(k)$, then the orbit $\mathbf{G}v$ is a smooth k-$\mathbf{G}$-variety and the orbit map $\alpha : \mathbf{G} \to \mathbf{G}v$ is a k-morphism.*

PROOF. Let $v \in \mathbf{V}$ be a point and set $\mathbf{W} = \overline{\mathbf{G}v}$. Consider the $\mathbf{G}$-equivariant morphism $f : \mathbf{G} \to \mathbf{W} : g \mapsto gv$. By Theorem 3.3, there exists a nonempty Zariski open set $\mathbf{U}$ of $\mathbf{W}$ such that $\mathbf{U} \subset \mathbf{G}v$. Then $\mathbf{G}v = \bigcup_{g \in \mathbf{G}} g\mathbf{U}$ is open in $\mathbf{W}$. Therefore, the orbit $\mathbf{G}v$ is locally closed in $\mathbf{V}$. Assume moreover that $v \in \mathbf{V}(k)$. Using Proposition 3.1, for every $\sigma \in \mathrm{Gal}(K/k)$, we have $(\mathbf{G}v)^\sigma = \mathbf{G}^\sigma \sigma(v) = \mathbf{G}v$ and so $\mathbf{G}v$ is a smooth k-$\mathbf{G}$-variety. Moreover, the orbit map $\alpha : \mathbf{G} \to \mathbf{G}v$ is a k-morphism. □

COROLLARY 3.14. *Let $\mathbf{G}$ and $\mathbf{H}$ be algebraic k-groups and $\varphi : \mathbf{G} \to \mathbf{H}$ a k-homomorphism. Then $\varphi(\mathbf{G}) < \mathbf{H}$ is a k-subgroup.*

PROOF. By Proposition 3.13, $\varphi(\mathbf{G})$ is open in $\overline{\varphi(\mathbf{G})}$. Since any open subgroup is also closed, it follows that $\varphi(\mathbf{G})$ is closed in $\overline{\varphi(\mathbf{G})}$ and so $\varphi(\mathbf{G}) = \overline{\varphi(\mathbf{G})}$. Therefore, $\varphi(\mathbf{G}) < \mathbf{H}$ is an algebraic subgroup. Moreover, the proof of Proposition 3.13 shows that $\varphi(\mathbf{G}) < \mathbf{H}$ is a k-subgroup. □

Let us point out that in Corollary 3.14, we always have $\varphi(\mathbf{G}(k)) \subset \varphi(\mathbf{G})(k)$ but in general $\varphi(\mathbf{G}(k)) \neq \varphi(\mathbf{G})(k)$. Indeed, consider $\varphi : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto z^2$. Then $\varphi(\mathbb{R}^*) = \mathbb{R}^*_+ \neq \mathbb{R}^*$.

PROPOSITION 3.15. *Let $\mathbf{G}$ be an algebraic k-group and $\mathbf{H} < \mathbf{G}$ a k-subgroup. The following assertions hold:*

(i) *The centralizer $\mathscr{Z}_{\mathbf{G}}(\mathbf{H}) < \mathbf{G}$ is a k-subgroup. In particular, the center $\mathscr{Z}(\mathbf{G}) < \mathbf{G}$ is a k-subgroup.*
(ii) *The normalizer $\mathscr{N}_{\mathbf{G}}(\mathbf{H}) < \mathbf{G}$ is a k-subgroup.*

PROOF. Consider the k-$\mathbf{G}$-variety $\mathbf{V} = \mathbf{G}$ with the conjugation action $\mathbf{G} \curvearrowright \mathbf{G}$. For every $x \in \mathbf{G}$, consider the orbit map $\alpha_x : \mathbf{G} \to \mathbf{G} : g \mapsto gxg^{-1}$, which is a morphism of varieties.

(i) By definition, we have

$$\mathscr{Z}_{\mathbf{G}}(\mathbf{H}) = \{g \in \mathbf{G} \mid \forall h \in \mathbf{H}, \alpha_h(g) = h\} = \bigcap_{h \in \mathbf{H}} \alpha_h^{-1}(\{h\}).$$

It follows that $\mathscr{Z}_{\mathbf{G}}(\mathbf{H}) < \mathbf{G}$ is Zariski closed. Using Proposition 3.1 and since $\mathbf{H}$ is defined over k, for every $\sigma \in \mathrm{Gal}(K/k)$, every $g \in \mathscr{Z}_{\mathbf{G}}(\mathbf{H})$ and every $h \in \mathbf{H}$, we have $\sigma(g)h = \sigma(g\sigma^{-1}(h)) = \sigma(\sigma^{-1}(h)g) = h\sigma(g)$ and so $\sigma(g) \in \mathscr{Z}_{\mathbf{G}}(\mathbf{H})$. This implies that $\mathscr{Z}_{\mathbf{G}}(\mathbf{H})$ is a k-subgroup.

(ii) Using the descending chain condition, we have

$$\mathscr{N}_{\mathbf{G}}(\mathbf{H}) = \{g \in \mathbf{G} \mid \forall h \in \mathbf{H}, \alpha_h(g) \in \mathbf{H}\} = \bigcap_{h \in \mathbf{H}} \alpha_h^{-1}(\mathbf{H}).$$

It follows that $\mathscr{N}_{\mathbf{G}}(\mathbf{H}) < \mathbf{G}$ is Zariski closed. Using Proposition 3.1 and since $\mathbf{H}$ is defined over k, for every $\sigma \in \mathrm{Gal}(K/k)$, every $g \in \mathscr{N}_{\mathbf{G}}(\mathbf{H})$ and every $h \in \mathbf{H}$, we have $\sigma(g)h\sigma(g^{-1}) = \sigma(g\sigma^{-1}(h)g^{-1}) \in \mathbf{H}$. This implies that $\mathscr{N}_{\mathbf{G}}(\mathbf{H})$ is a k-subgroup. □

Note that if $\mathbf{G}$ is a connected algebraic group, then any finite (algebraic) normal subgroup is necessarily central. Indeed, let $\mathbf{H} \lhd \mathbf{G}$ be a finite (algebraic) normal subgroup. Then $\mathscr{Z}_{\mathbf{G}}(\mathbf{H}) < \mathbf{G}$ is a finite index algebraic subgroup by Proposition 3.15. Since $\mathbf{G}$ is connected, we have $\mathbf{G} = \mathscr{Z}_{\mathbf{G}}(\mathbf{H})$ by Proposition 3.9 and so $\mathbf{H} < \mathscr{Z}(\mathbf{G})$.

The next result enables us to define the notion of homogeneous space in the setting of algebraic k-groups.

THEOREM 3.16 (Chevalley). *Let $\mathbf{G}$ be an algebraic k-group and $\mathbf{H} < \mathbf{G}$ a k-subgroup. Then there exists a k-representation $\pi : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ and a point $x \in \mathrm{P}(\mathbf{V}(k))$ such that $\mathbf{H} = \mathrm{Stab}_{\mathbf{G}}(x)$.*

In particular, the homogeneous space $\mathbf{G}/\mathbf{H}$ has a natural structure of smooth quasiprojective algebraic k-$\mathbf{G}$-variety. Moreover, the canonical projection $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is a $\mathbf{G}$-equivariant k-morphism such that $\mathbf{H} = \pi(e) \in (\mathbf{G}/\mathbf{H})(k)$.

PROOF. For every $m \geq 1$, set $K^m[\mathbf{G}] = \{P \in K[\mathbf{G}] \mid \deg(P) \leq m\}$. Then $K^m[\mathbf{G}]$ is a finite dimensional k-subspace of $K[\mathbf{G}]$. Since $K[\mathbf{G}]$ is Noetherian, the vanishing ideal $I(\mathbf{H})$ is finitely generated. Then there exists $m \geq 1$ such that $I^m(\mathbf{H}) = K^m[\mathbf{G}] \cap I(\mathbf{H})$ generates $I(\mathbf{H})$. Consider the well-defined k-representation $\rho : \mathbf{G} \to \mathrm{GL}(K^m[\mathbf{G}])$ given by $(\rho(g)P)(h) = P(hg)$ for all $g, h \in \mathbf{G}$ and all $P \in K^m[\mathbf{G}]$.

We claim that $g \in \mathbf{H}$ if and only if $\rho(g)(I^m(\mathbf{H})) = I^m(\mathbf{H})$. Indeed, it is clear that if $g \in \mathbf{H}$, then $\rho(g)(I^m(\mathbf{H})) = I^m(\mathbf{H})$. Conversely, assume that $\rho(g)(I^m(\mathbf{H})) = I^m(\mathbf{H})$. In particular, for every $P \in I^m(\mathbf{H})$, we have $P(g) = \rho(g)(P)(e) = 0$. This implies that $g \in \mathbf{H}$.

Denote by $p = \dim_K(I^m(\mathbf{H}))$. Consider the pth exterior algebra $\mathbf{V} = \bigwedge^p(K^m[\mathbf{G}])$, which is a finite dimensional k-vector space. Then $\mathbf{W} = \bigwedge^p(I^m(\mathbf{H}))$ is a one dimensional k-subspace of $\mathbf{V}$ and the natural map $\pi = \bigwedge^p \rho : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ is a k-representation. Moreover, we have

$$\mathbf{H} = \{g \in \mathbf{G} \mid \pi(g)(\mathbf{W}) = \mathbf{W}\}.$$

Then we can take $x = \mathbf{W}(k) \in \mathrm{P}(\mathbf{V}(k))$ and we have $\mathbf{H} = \mathrm{Stab}_{\mathbf{G}}(x)$.

We may identify $\mathbf{G}/\mathbf{H}$ with $\mathbf{G}x$. Since the orbit $\mathbf{G}x$ is open in its closure $\overline{\mathbf{G}x}$, it follows that $\mathbf{G}/\mathbf{H}$ has a natural structure of smooth quasiprojective algebraic k-$\mathbf{G}$-variety. Moreover, the canonical projection $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is a $\mathbf{G}$-equivariant k-morphism such that $\mathbf{H} = \pi(e) \in (\mathbf{G}/\mathbf{H})(k)$. $\square$

Whenever $\mathbf{H}$ is an algebraic k-group, we denote by $X_k(\mathbf{H})$ the abelian group of all k-regular characters $\chi : \mathbf{H} \to \mathbf{G}_m$. In Theorem 3.16, in case $X_k(\mathbf{H}) = \{1\}$, we can choose a point $v \in \mathbf{V}(k)$ such that $\mathbf{H} = \{h \in \mathbf{G} \mid \pi(g)v = v\}$.

Next, we record the following useful universal property of the homogeneous space $\mathbf{G}/\mathbf{H}$.

PROPOSITION 3.17. *Let $\mathbf{G}$ be an algebraic k-group and $\mathbf{V}$ an algebraic k-$\mathbf{G}$-variety. For every $v \in \mathbf{V}(k)$, the orbit $\mathbf{G}v$ is a smooth k-$\mathbf{G}$-variety,* *the stabilizer* $\mathbf{H} = \mathrm{Stab}_{\mathbf{G}}(v) < \mathbf{G}$ *is a* k*-subgroup and there exists a unique* $\mathbf{G}$*-equivariant* k*-isomorphism* $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{G}v$ *such that* $\alpha = \beta \circ \pi$*, where* $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ *is the canonical projection and* $\alpha : \mathbf{G} \to \mathbf{G}v$ *is the orbit map.*

PROOF. Using Proposition 3.1, for every $\sigma \in \mathrm{Gal}(K/k)$ and every $h \in \mathbf{H}$, we have $\sigma(h)v = \sigma(hv) = \sigma(v) = v$ and so $\mathbf{H}^\sigma = \mathbf{H}$. Thus, $\mathbf{H} < \mathbf{G}$ is a k-subgroup. Consider the product k-$\mathbf{G}$-variety $\mathbf{W} = \mathbf{G}/\mathbf{H} \times \mathbf{G}v$. Then the orbit map $\Theta = \pi \times \alpha : \mathbf{G} \to \mathbf{W} : g \mapsto (g\mathbf{H}, gv)$ is a k-morphism and the orbit $\Theta(\mathbf{G}) = \mathbf{G} \cdot (\mathbf{H}, v)$ is a smooth k-$\mathbf{G}$-variety. Denote by $p_1 : \mathbf{W} \to \mathbf{G}/\mathbf{H}$ (resp. $p_2 : \mathbf{W} \to \mathbf{G}v$) the projection on the first (resp. second) coordinate. Since $p_1 \circ \Theta = \pi$ and since π is surjective, $p_1 : \mathbf{W} \to \mathbf{G}/\mathbf{H}$ is surjective. Moreover, for all $g, h \in \mathbf{G}$, if $p_1(\Theta(g)) = p_1(\Theta(h))$, then $\pi(g) = \pi(h)$ which implies that $g^{-1}h \in \mathbf{H}$ and so $\Theta(g) = \Theta(h)$. It follows that $p_1 : \Theta(\mathbf{G}) \to \mathbf{G}/\mathbf{H}$ is a bijective k-morphism. By homogeneity, Corollary 3.5 implies that $p_1 : \Theta(\mathbf{G}) \to \mathbf{G}/\mathbf{H}$ is a k-isomorphism. Then $\beta = p_2 \circ (p_1)^{-1} : \mathbf{G}/\mathbf{H} \to \mathbf{G}v$ is a k-morphism such that $\alpha = \beta \circ \pi$. Observe that since $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is surjective, the $\mathbf{G}$-equivariant k-morphism $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{G}v$ is necessarily unique. By injectivity and homogeneity, Corollary 3.5 implies that $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{G}v$ is a k-isomorphism. □

In the case when the k-subgroup $\mathbf{H} \lhd \mathbf{G}$ is normal, we show that the k-$\mathbf{G}$-variety $\mathbf{G}/\mathbf{H}$ is a linear algebraic k-group.

THEOREM 3.18. *Let* $\mathbf{G}$ *be an algebraic* k*-group and* $\mathbf{H} \lhd \mathbf{G}$ *a normal* k*-subgroup. Then the* k*-*$\mathbf{G}$*-variety* $\mathbf{G}/\mathbf{H}$ *is a linear algebraic* k*-group.*

PROOF. Since $\mathbf{H} \lhd \mathbf{G}$ is normal, we may consider the action $\mathbf{G} \curvearrowright X_k(\mathbf{H})$ defined by $(g\chi)(h) = \chi(g^{-1}hg)$ for every $g \in \mathbf{G}$, every $h \in \mathbf{H}$ and every $\chi \in X_k(\mathbf{H})$. Keep the same notation as in the proof of Theorem 3.16. We have a k-representation $\pi : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ and a one-dimensional k-subspace $\mathbf{W} \subset \mathbf{V}$ such that $\mathbf{H} = \{g \in \mathbf{G} \mid \pi(g)(\mathbf{W}) = \mathbf{W}\}$. Choose a nonzero vector $w \in \mathbf{W}(k)$ and denote by $\chi_{\mathbf{W}} \in X_k(\mathbf{H})$ the unique k-character such that $\pi(h)w = \chi_{\mathbf{W}}(h)w$ for every $h \in \mathbf{H}$. Denote by $Y = \mathbf{G} \cdot \chi_{\mathbf{W}} \subset X_k(\mathbf{H})$ the $\mathbf{G}$-orbit of $\chi_{\mathbf{W}}$ in $X_k(\mathbf{W})$.

For every $\chi \in Y$, set $\mathbf{V}_\chi = \{v \in \mathbf{V} \mid \forall g \in \mathbf{H},\ \pi(g)v = \chi(g)v\}$. The sum $\sum_{\chi \in Y} \mathbf{V}_\chi$ is direct and globally invariant under $\pi(\mathbf{G})$. Upon replacing $\mathbf{V}$ by $\bigoplus_{\chi \in Y} \mathbf{V}_\chi$, we may assume that $\mathbf{V} = \bigoplus_{\chi \in Y} \mathbf{V}_\chi$. Observe that $\mathbf{W} \subset \mathbf{V}_{\chi_{\mathbf{W}}}$.

Consider the adjoint k-representation $\mathrm{Ad} : \mathrm{GL}(\mathbf{V}) \to \mathrm{GL}(\mathrm{End}(\mathbf{V})) : g \mapsto (u \mapsto gug^{-1})$. Define the k-subspace $\mathbf{A} = \bigoplus_{\chi \in Y} \mathrm{End}(\mathbf{V}_\chi) \subset \mathrm{End}(\mathbf{V})$ of all endomorphisms preserving the direct sum $\bigoplus_{\chi \in Y} \mathbf{V}_\chi$. Then $\mathbf{A} \subset \mathrm{End}(\mathbf{V})$ is globally invariant under $(\mathrm{Ad} \circ \pi)(\mathbf{G})$ and we denote by $\Psi = \mathrm{Ad} \circ \pi : \mathbf{G} \to \mathrm{GL}(\mathbf{A})$ the corresponding k-representation. We claim that $\mathbf{H} = \ker(\Psi)$. Indeed, let $h \in \mathbf{H}$. Since $\pi(h)$ acts by scalar multiplication on each k-subspace $\mathbf{V}_\chi$ for $\chi \in Y$, it follows that $\Psi(h) = 1$. Conversely, let $g \in \mathbf{G}$ be such $\Psi(g) = 1$. Then $\pi(g)$ is central in $\mathbf{A}$ and so $\pi(g)$ acts by scalar multiplication on each k-subspace $\mathbf{V}_\chi$ for $\chi \in Y$. In particular, we have $\pi(g)(\mathbf{W}) = \mathbf{W}$ and so $g \in \mathbf{H}$.

Then $\Psi(\mathbf{G}) < \mathrm{GL}(\mathbf{A})$ is a k-subgroup. By Proposition 3.17, there exists a unique $\mathbf{G}$-equivariant k-isomorphism $\beta : \mathbf{G}/\mathbf{H} \to \Psi(\mathbf{G})$ such that $\Psi = \beta \circ \pi$. Moreover, $\beta : \mathbf{G}/\mathbf{H} \to \Psi(\mathbf{G})$ is a group homomorphism. This shows that $\mathbf{G}/\mathbf{H}$ is a linear algebraic k-group. □

3. Stabilizers and tameness of algebraic actions

In this section, we assume that k is a *local* (i.e. nondiscrete locally compact) field of characteristic zero. It is known that k is either $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ for some prime $p \in \mathscr{P}$.

For any algebraic k-variety $\mathbf{V}$, the set $\mathbf{V}(k)$ of its k-points is endowed with a natural topology induced from the topology of the local field k. Then $\mathbf{V}(k)$ is a Hausdorff locally compact second countable topological space. If the algebraic k-variety $\mathbf{V}$ is moreover smooth, then $\mathbf{V}(k)$ has a natural structure of smooth k-analytic manifold. In that case, for every $v \in \mathbf{V}(k)$, the space $\mathscr{T}_v(\mathbf{V})_k$ can be identified with the tangent space $\mathscr{T}_v(\mathbf{V}(k))$ of the smooth k-analytic manifold $\mathbf{V}(k)$ at the point $v \in \mathbf{V}(k)$.

For any algebraic k-group $\mathbf{G}$, the group $\mathbf{G}(k)$ of its k-points has a natural structure of k-analytic Lie group. In particular, if $k = \mathbb{R}$, then $\mathbf{G}(\mathbb{R})$ is a real Lie group. Moreover, the space $\mathrm{Lie}(\mathbf{G})_k$ can be identified with the Lie algebra $\mathrm{Lie}(\mathbf{G}(k))$ of the k-analytic Lie group $\mathbf{G}(k)$.

Proposition 3.19. *Let $\mathbf{G}$ be a Zariski connected algebraic k-group. Then $\mathbf{G}(k)$ is Zariski dense in $\mathbf{G}$.*

Proof. Denote by $\mathbf{H}$ the Zariski closure of $\mathbf{G}(k)$ in $\mathbf{G}$. Then $\mathbf{H} < \mathbf{G}$ is a k-subgroup by Proposition 3.2. Moreover, we have $\mathrm{Lie}(\mathbf{H}) = \mathrm{Lie}(\mathbf{G})$. Since $\mathbf{G}$ is Zariski connected, it follows that $\mathbf{H} = \mathbf{G}$ (see [**Bo91**, 7.1]). □

Let us point that even though $\mathbf{G}$ is Zariski connected, the k-analytic Lie group $\mathbf{G}(k)$ need not be connected. More precisely, if $k = \mathbb{C}$, then $\mathbf{G}(k)$ is connected for the analytic topology. If $k = \mathbb{R}$, then the connected component $\mathbf{G}(\mathbb{R})^0$ of the identity element has finite index in $\mathbf{G}(\mathbb{R})$.

3.1. Stabilizers and tameness of $G \curvearrowright V$. In this subsection, we use the following notation. Let $\mathbf{G}$ be a connected algebraic k-group and $\mathbf{V}$ an algebraic k-$\mathbf{G}$-variety. Set $G = \mathbf{G}(k)$ and $V = \mathbf{V}(k)$. For every $v \in V$, the $\mathbf{G}$-orbit $\mathbf{G}v$ is a smooth k-$\mathbf{G}$-variety, the stabilizer $\mathbf{H} = \mathrm{Stab}_{\mathbf{G}}(v) < \mathbf{G}$ is a k-subgroup and there exists a unique $\mathbf{G}$-equivariant k-isomorphism $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{G}v$ such that $\alpha = \beta \circ \pi$, where $\alpha : \mathbf{G} \to \mathbf{G}v$ is the orbit map and $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is the canonical $\mathbf{G}$-equivariant k-regular projection.

Theorem 3.20. *Keep the same notation as above. For every $v \in V$, the G-orbit Gv is locally closed in V for the analytic topology. Moreover, letting $H = \mathbf{H}(k) = \mathrm{Stab}_G(v)$, the restriction $\beta|_{G/H} : G/H \to Gv$ is a homeomorphism, when G/H is endowed with the quotient topology and $Gv \subset V$ is endowed with the relative topology.*

In particular, Theorem 3.20 implies that the Borel action $G \curvearrowright V$ is tame and the quotient Borel space $G\backslash V$ is standard.

Proof of Theorem 3.20. Since the orbit map $\alpha : \mathbf{G} \to \mathbf{G}v$ is surjective, Corollary 3.5 implies that the differential $\mathrm{d}_g\alpha$ is surjective on a nonempty Zariski open set $\mathbf{U} \subset \mathbf{G}$. By homogeneity, the differential $\mathrm{d}_g\alpha$ is surjective everywhere on $\mathbf{G}$. In particular, using the identification $\mathscr{T}_e(\mathbf{G}) = \mathrm{Lie}(\mathbf{G})$, we have that $\mathrm{d}_e\alpha : \mathrm{Lie}(\mathbf{G}) \to \mathscr{T}_v(\mathbf{G}v)$ is surjective. Then

$$\dim_K(\mathrm{Lie}(\mathbf{G})) = \dim_K(\mathscr{T}_v(\mathbf{G}v)) + \dim_K(\ker(\mathrm{d}_e\alpha)).$$

This further implies that

$$\dim_k(\mathrm{Lie}(G)) = \dim_k(\mathscr{T}_v((\mathbf{G}v)(k))) + \dim_k(\ker(\mathrm{d}_e\alpha)(k))$$

and so the k-linear map $\mathrm{d}_e\alpha|_{\mathrm{Lie}(G)} : \mathrm{Lie}(G) \to \mathscr{T}_v((\mathbf{G}v)(k))$ is surjective. By the submersion theorem (see [Se65, Chapter III, Theorem, pp. 85]), $\alpha|_G : G \to (\mathbf{G}v)(k)$ is open on a neighborhood of the identity element. This further implies that all the G-orbits are open in $(\mathbf{G}v)(k)$. Therefore, all the G-orbits are both open and closed in $(\mathbf{G}v)(k)$. In particular, the orbit Gv is both open and closed in $(\mathbf{G}v)(k)$.

By Theorem 3.3, $\mathbf{G}v$ is Zariski locally closed in $\mathbf{V}$. In particular, it follows that $(\mathbf{G}v)(k)$ is locally closed in $V = \mathbf{V}(k)$ for the analytic topology. Therefore, we infer that Gv is locally closed in V. The conclusion of the theorem follows from Proposition 2.12. □

For any algebraic k-group $\mathbf{G}$ and any k-subgroup $\mathbf{H} < \mathbf{G}$, we consider the algebraic k-$\mathbf{G}$-variety $\mathbf{V} = \mathbf{G}/\mathbf{H}$ and the canonical $\mathbf{G}$-equivariant k-regular projection $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ so that $\mathbf{H} = \pi(e) \in (\mathbf{G}/\mathbf{H})(k)$. The proof of Theorem 3.20 shows that the $\mathbf{G}(k)$-orbit $\mathbf{G}(k)\pi(e)$ is open and closed in $(\mathbf{G}/\mathbf{H})(k)$. Since $\mathbf{H}(k) = \mathrm{Stab}_{\mathbf{G}(k)}(\pi(e))$, we may identify the $\mathbf{G}(k)$-orbit $\mathbf{G}(k)\pi(e)$ with $\mathbf{G}(k)/\mathbf{H}(k)$ and we have a natural continuous injective mapping $\iota : \mathbf{G}(k)/\mathbf{H}(k) \hookrightarrow (\mathbf{G}/\mathbf{H})(k)$. Let us point out that in general, we have $\iota(\mathbf{G}(k)/\mathbf{H}(k)) \neq (\mathbf{G}/\mathbf{H})(k)$. In what follows, we regard $\mathbf{G}(k)/\mathbf{H}(k) \subset (\mathbf{G}/\mathbf{H})(k)$ as an open and closed subset.

Let (X, ν) be a standard probability space. We denote by $\mathscr{V} = \mathrm{L}^0(X, V)$ the space of all ν-equivalence classes of measurable maps $\psi : X \to V$. Endowed with the topology of convergence in measure, $\mathscr{V}$ is a Polish space. Consider the continuous action $G \curvearrowright \mathscr{V}$ defined by $(g\psi)(x) = g\psi(x)$ for every $g \in G$ and every $\psi \in \mathscr{V}$. We obtain the following generalization of Theorem 3.20.

Theorem 3.21. *Keep the same notation as above. For every $\psi \in \mathscr{V}$, the orbit $G\psi$ is locally closed in $\mathscr{V}$ and there exists a k-subgroup $\mathbf{H}_\psi < \mathbf{G}$ such that $\mathrm{Stab}_G(\psi) = \mathbf{H}_\psi(k)$.*

Proof. Let $\psi \in \mathscr{V}$. In order to show that the orbit $G\psi$ is locally closed in $\mathscr{V}$, using Proposition 2.12, it suffices to show that the map $G\psi \to G/\mathrm{Stab}_G(\psi) : g\psi \mapsto g\,\mathrm{Stab}_G(\psi)$ is continuous. Using the fact that convergence in measure for a sequence implies convergence almost everywhere for a subsequence, it suffices to prove that for any sequence $(g_n)_n$ in G such that $g_n\psi \to \psi$ ν-almost everywhere, $g_n \operatorname{Stab}_G(\psi) \to \operatorname{Stab}_G(\psi)$ in $G/\operatorname{Stab}_G(\psi)$. Let $X_1 \subset X$ be a conull measurable subset and $(g_n)_n$ a sequence in G such that $g_n\psi(x) \to \psi(x)$ for every $x \in X_1$. Fix a countable dense subgroup $\Lambda < \operatorname{Stab}_G(\psi)$. It is clear that $g \in \operatorname{Stab}_G(\psi)$ if and only if $g \in \operatorname{Stab}_G(\psi(x))$ for ν-almost every $x \in X$. Since $\Lambda < \operatorname{Stab}_G(\psi)$ is countable, there exists a conull measurable subset $X_0 \subset X_1 \subset X$ such that $\Lambda \subset \bigcap_{x\in X_0} \operatorname{Stab}_G(\psi(x))$. Since the latter group is closed, it follows that $\operatorname{Stab}_G(\psi) \subset \bigcap_{x\in X_0} \operatorname{Stab}_G(\psi(x))$. Thus, we have $\operatorname{Stab}_G(\psi) = \bigcap_{x\in X_0} \operatorname{Stab}_G(\psi(x))$. For every $x \in X_0$, set $\mathbf{H}_{\psi(x)} = \operatorname{Stab}_{\mathbf{G}}(\psi(x))$ which is a k-subgroup by Proposition 3.17. By the descending chain condition, there exist $x_1, \dots, x_p \in X$ such that $\bigcap_{x\in X_0} \mathbf{H}_{\psi(x)} = \bigcap_{i=1}^{p} \mathbf{H}_{\psi(x_i)}$. Set $\mathbf{H}_\psi = \bigcap_{i=1}^{p} \mathbf{H}_{\psi(x_i)}$, which is a k-subgroup of $\mathbf{G}$. Then we have

$$\operatorname{Stab}_G(\psi) = \mathbf{H}_\psi(k) = \bigcap_{i=1}^{p} \mathbf{H}_{\psi(x_i)}(k) = \bigcap_{i=1}^{p} \operatorname{Stab}_G(\psi(x_i)).$$

Set $Y = \prod_{i=1}^{p} G/\operatorname{Stab}_G(\psi(x_i))$ and $y = (\operatorname{Stab}_G(\psi(x_i)))_i \in Y$. Observe that $\operatorname{Stab}_G(y) = \bigcap_{i=1}^{p} \operatorname{Stab}_G(\psi(x_i)) = \operatorname{Stab}_G(\psi)$. Moreover, Y is homeomorphic to $\prod_{i=1}^{p} G\psi(x_i)$, which is locally closed in $\prod_{i=1}^{p}(\mathbf{G}\psi(x_i))(k)$. By applying Theorem 3.20 to the algebraic k-$\mathbf{G}$-variety $\prod_{i=1}^{p} \mathbf{G}\psi(x_i)$, we obtain that the continuous action $G \curvearrowright Y$ has locally closed orbits and so the map $Gy \to G/\operatorname{Stab}_G(y) : gy \mapsto g\operatorname{Stab}_G(y)$ is continuous by Proposition 2.12. Since $g_n\psi(x_i) \to \psi(x_i)$ for every $i \in \{1, \dots, p\}$, Proposition 2.12 and Theorem 3.20 imply that $g_n \operatorname{Stab}_G(\psi(x_i)) \to \operatorname{Stab}_G(\psi(x_i))$ for every $i \in \{1, \dots, p\}$. This further implies that $g_n y \to y$ and so $g_n \operatorname{Stab}_G(\psi) = g_n \operatorname{Stab}_G(y) \to \operatorname{Stab}_G(y) = \operatorname{Stab}_G(\psi)$. □

3.2. Stabilizers and tameness of $G \curvearrowright \operatorname{Prob}(V)$. For a locally compact second countable group L, a standard Borel space Z and a Borel action $L \curvearrowright Z$, we denote by $\operatorname{Prob}(Z)^L$ the standard Borel space of all L-invariant Borel probability measures on Z. As usual, we simply write $\operatorname{Prob}(Z) = \operatorname{Prob}(Z)^{\{e\}}$.

Definition 3.22. Let $\mathbf{G}$ be an algebraic k-group and set $G = \mathbf{G}(k)$. A closed subgroup $L < G$ is said to be *almost algebraic* if there exists a k-subgroup $\mathbf{H} < \mathbf{G}$ such that $H = \mathbf{H}(k)$ sits as a cocompact closed normal subgroup in L.

The first main result of this subsection is due to Bader–Duchesne–Lécureux (see [**BDL14**, Proposition 1.9]). It is a generalization of Shalom's result [**Sh97**, Theorem 1.1].

Theorem 3.23. *Let $\mathbf{G}$ be an algebraic k-group and set $G = \mathbf{G}(k)$. Let $L < G$ be a closed subgroup that is Zariski dense in $\mathbf{G}$.*

Then there exists a normal k-subgroup $\mathbf{N} \lhd \mathbf{G}$ such that the image of L in $(\mathbf{G}/\mathbf{N})(k)$ is precompact. Moreover, for every algebraic k-$\mathbf{G}$-variety $\mathbf{V}$ and *every Borel probability measure* $\mu \in \mathrm{Prob}(\mathbf{V}(k))^L$, *we have* $\mu = \mu|_{\mathbf{V}^{\mathbf{N}} \cap \mathbf{V}(k)}$, *that is,* μ *is supported on the subset* $\mathbf{V}^{\mathbf{N}} \cap \mathbf{V}(k)$ *of* $\mathbf{N}$*-fixed points.*

PROOF. Consider the set $\mathscr{A}$ consisting of all algebraic subgroups $\mathbf{H} < \mathbf{G}$ for which there exists $g \in \mathbf{G}$ such that $g\mathbf{H}g^{-1} < \mathbf{G}$ is a k-subgroup and such that $\mathrm{Prob}((\mathbf{G}/g\mathbf{H}g^{-1})(k))^L \neq \emptyset$. Note that $\mathbf{G} \in \mathscr{A}$ and so $\mathscr{A} \neq \emptyset$. Since the ring $K[\mathbf{G}]$ is Noetherian, $\mathscr{A}$ contains a minimal element $\mathbf{H}_{\min} < \mathbf{G}$. Choose $h \in \mathbf{G}$ such that $\mathbf{H}_0 = h\mathbf{H}_{\min}h^{-1} < \mathbf{G}$ is a k-subgroup and $\mathrm{Prob}((\mathbf{G}/\mathbf{H}_0)(k))^L \neq \emptyset$. Set $\mathbf{V}_0 = \mathbf{G}/\mathbf{H}_0$ and choose $\mu_0 \in \mathrm{Prob}(\mathbf{V}_0(k))^L$. Firstly, we prove the following claim.

CLAIM 3.24. $\mathbf{H}_0$ is a normal k-subgroup of $\mathbf{G}$.

PROOF OF CLAIM 3.24. Denote by $\mathbf{N} = \mathscr{N}_{\mathbf{G}}(\mathbf{H}_0)$ the normalizer of $\mathbf{H}_0$ in $\mathbf{G}$. Then $\mathbf{N} < \mathbf{G}$ is a k-subgroup by Proposition 3.15. By contradiction, assume that $\mathbf{N} \neq \mathbf{G}$. Consider

$$\mathbf{U} = \left\{(x\mathbf{H}_0, y\mathbf{H}_0) \in \mathbf{V}_0 \times \mathbf{V}_0 \mid y^{-1}x \notin \mathbf{N}\right\}.$$

Observe that $\mathbf{U} \subset \mathbf{V}_0 \times \mathbf{V}_0$ is a nonempty Zariski open set that is invariant under the diagonal action $\mathbf{G} \curvearrowright \mathbf{V}_0 \times \mathbf{V}_0$. Indeed, its complement $(\mathbf{V}_0 \times \mathbf{V}_0) \setminus \mathbf{U}$ is the inverse image of the diagonal $\{(g\mathbf{N}, g\mathbf{N}) \mid g\mathbf{N} \in \mathbf{G}/\mathbf{N}\}$ under the canonical k-morphism $\mathbf{G}/\mathbf{H}_0 \times \mathbf{G}/\mathbf{H}_0 \to \mathbf{G}/\mathbf{N} \times \mathbf{G}/\mathbf{N}$. Moreover, $\mathbf{U}$ is defined over k.

Denote by $B \subset (\mathbf{V}_0 \times \mathbf{V}_0)(k)$ the topological support of $\mu_0 \otimes \mu_0 \in \mathrm{Prob}((\mathbf{V}_0 \times \mathbf{V}_0)(k))^{L \times L}$ and by $\mathbf{W}$ the Zariski closure of B in $\mathbf{V}_0 \times \mathbf{V}_0$. Since B is $(L \times L)$-invariant and since $L \times L$ is Zariski dense in $\mathbf{G} \times \mathbf{G}$, it follows that $\mathbf{W} \subset \mathbf{V}_0 \times \mathbf{V}_0$ is $\mathbf{G} \times \mathbf{G}$-invariant and so $\mathbf{W} = \mathbf{V}_0 \times \mathbf{V}_0$. Since $\mathbf{U} \subset \mathbf{V}_0 \times \mathbf{V}_0$ is Zariski open, we have that $(\mu_0 \otimes \mu_0)(\mathbf{U}(k)) > 0$. Indeed, otherwise we would have $B \subset \mathbf{V}(k) \setminus \mathbf{U}(k)$ and so $\mathbf{W} \subset \mathbf{V} \setminus \mathbf{U} \neq \mathbf{V}_0 \times \mathbf{V}_0$, a contradiction. We regard $(\mu_0 \otimes \mu_0)|_{\mathbf{U}(k)}$ as a nonzero L-invariant measure for the diagonal action and we set

$$\eta = \frac{1}{(\mu_0 \otimes \mu_0)(\mathbf{U}(k))}(\mu_0 \otimes \mu_0)|_{\mathbf{U}(k)} \in \mathrm{Prob}(\mathbf{U}(k))^L.$$

By Theorem 3.20, the Borel action $G \curvearrowright \mathbf{U}(k)$ is tame and Corollary A.6 implies that there exists $u \in \mathbf{U}(k)$ such that $\mathrm{Prob}(Gu)^L \neq \emptyset$. By Theorem 3.20, denote by $\mathbf{H} = \mathrm{Stab}_{\mathbf{G}}(u)$ the stabilizer k-subgroup and by $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{G}u$ the k-isomorphism. Letting $H = \mathbf{H}(k)$, we have that $\beta|_{G/H} : G/H \to Gu$ is a homeomorphism. Regarding $G/H \hookrightarrow (\mathbf{G}/\mathbf{H})(k)$, it follows that $\mathrm{Prob}((\mathbf{G}/\mathbf{H})(k))^L \neq \emptyset$. Write $u = (x\mathbf{H}_0, y\mathbf{H}_0) \in \mathbf{V}_0(k) \times \mathbf{V}_0(k)$. Since $\mathbf{H} = \mathrm{Stab}_{\mathbf{G}}(u)$, we have $\mathbf{H} < x\mathbf{H}_0x^{-1}$ and so $h^{-1}x^{-1}\mathbf{H}xh < \mathbf{H}_{\min}$. By minimality of $\mathbf{H}_{\min}$, we obtain $h^{-1}x^{-1}\mathbf{H}xh = \mathbf{H}_{\min}$ and so $x^{-1}\mathbf{H}x = \mathbf{H}_0$. Likewise, we have $y^{-1}\mathbf{H}y = \mathbf{H}_0$. This implies that $y^{-1}x \in \mathbf{N}$, a contradiction. Thus $\mathbf{H}_0 \lhd \mathbf{G}$ is a normal k-subgroup. In particular, we have $\mathbf{H}_0 = \mathbf{H}_{\min}$. $\square$

Since $\mathbf{H}_0 \lhd \mathbf{G}$ is a normal k-subgroup by Claim 3.24, we simply write $\mathbf{H}_0 = \mathbf{N}$. We fix $\mu_N \in \mathrm{Prob}((\mathbf{G}/\mathbf{N})(k))^L$.

Secondly, we show that the image of L in $(\mathbf{G}/\mathbf{N})(k)$ is precompact. Indeed, set $S = (\mathbf{G}/\mathbf{N})(k)$ and denote by T the closure of the image of L in S. By Theorem 2.10 and considering the left translation action $T \curvearrowright S$, the quotient space $T\backslash S$ is a Hausdorff locally compact second countable space and so $T\backslash S$ is a standard Borel space. By continuity and density and since $\mu_N \in \mathrm{Prob}(S)^L$, we have $\mu_N \in \mathrm{Prob}(S)^T$. Then Corollary A.6 implies that there exists $s \in S$ and $\eta \in \mathrm{Prob}(Ts)^T$. Denote by $\mu = R_{s*}\eta \in \mathrm{Prob}(T)^T$ the pushforward measure of η by the right translation by s^{-1}. Then T is a locally compact group that carries a Haar probability measure and so T is compact by Proposition 1.6. This shows that the image of L in $(\mathbf{G}/\mathbf{N})(k)$ is precompact.

Thirdly, let $\mathbf{V}$ be a k-$\mathbf{G}$-variety and $\mu \in \mathrm{Prob}(\mathbf{V}(k))^L$ an L-invariant Borel probability measure on $\mathbf{V}(k)$. We need to show that $\mu = \mu|_{\mathbf{V}^\mathbf{N} \cap \mathbf{V}(k)}$. By contradiction, assume that $\mu \neq \mu|_{\mathbf{V}^\mathbf{N} \cap \mathbf{V}(k)}$. Denote by $\mathbf{W}$ the Zariski closure of $\mathbf{V}^\mathbf{N} \cap \mathbf{V}(k)$ in $\mathbf{V}$. Then $\mathbf{W}$ is defined over k by Proposition 3.2 and since $\mathbf{N} \lhd \mathbf{G}$, $\mathbf{W}$ is $\mathbf{G}$-invariant. By definition, $\mathbf{N}$ acts trivially on $\mathbf{W}$ and so $\mathbf{W}(k) = \mathbf{V}^\mathbf{N} \cap \mathbf{V}(k)$. Consider the k-$\mathbf{G}$-variety $\mathbf{U} = \mathbf{V}\backslash\mathbf{W}$. By assumption, we have $\mu(\mathbf{U}(k)) > 0$. Upon replacing $\mathbf{V}$ by $\mathbf{U}$ and considering $\frac{1}{\mu(\mathbf{U}(k))}\mu|_{\mathbf{U}(k)}$ on $\mathbf{U}(k)$, we may assume that $\mathbf{V} = \mathbf{U}$, that is, $\mathbf{V}^\mathbf{N} \cap \mathbf{V}(k) = \emptyset$. Consider the k-$\mathbf{G}$-variety $\mathbf{G}/\mathbf{N} \times \mathbf{V}$ and observe that $\mu_N \otimes \mu \in \mathrm{Prob}((\mathbf{G}/\mathbf{N} \times \mathbf{V})(k))^L$. By Theorem 3.20, the Borel action $G \curvearrowright (\mathbf{G}/\mathbf{N} \times \mathbf{V})(k)$ is tame and Corollary A.6 implies that there exists $w \in (\mathbf{G}/\mathbf{N} \times \mathbf{V})(k)$ such that $\mathrm{Prob}(Gw)^L \neq \emptyset$. By Theorem 3.20, denote by $\mathbf{H} = \mathrm{Stab}_\mathbf{G}(w)$ the stabilizer k-subgroup and by $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{G}w$ the k-isomorphism. Letting $H = \mathbf{H}(k)$, we have that $\beta|_{G/H} : G/H \to Gw$ is a homeomorphism. Regarding $G/H \hookrightarrow (\mathbf{G}/\mathbf{H})(k)$, it follows that $\mathrm{Prob}((\mathbf{G}/\mathbf{H})(k))^L \neq \emptyset$. Write $w = (x\mathbf{N}, v) \in (\mathbf{G}/\mathbf{N} \times \mathbf{V})(k) = (\mathbf{G}/\mathbf{N})(k) \times \mathbf{V}(k)$. Since $\mathbf{H} = \mathrm{Stab}_\mathbf{G}(w)$ and since $\mathbf{N} \lhd \mathbf{G}$, we have $\mathbf{H} < x\mathbf{N}x^{-1} = \mathbf{N}$. By minimality of $\mathbf{N}$, we obtain $\mathbf{H} = \mathbf{N}$. This further implies that $v \in \mathbf{V}^\mathbf{N} \cap \mathbf{V}(k)$, a contradiction. This finishes the proof of the theorem. $\square$

Theorem 3.23 has several striking consequences. The first corollary deals with the structure of stabilizers of probability measures on algebraic varieties (see also [**Zi84**, Theorem 3.2.4]).

Corollary 3.25. *Let $\mathbf{G}$ be an algebraic k-group and $\mathbf{V}$ an algebraic k-$\mathbf{G}$-variety. For every $\mu \in \mathrm{Prob}(\mathbf{V}(k))$, the stabilizer $\mathrm{Stab}_{\mathbf{G}(k)}(\mu) < \mathbf{G}(k)$ is almost algebraic.*

Proof. Upon considering the Zariski closure of $L = \mathrm{Stab}_{\mathbf{G}(k)}(\mu)$ in $\mathbf{G}$, which is a k-subgroup of $\mathbf{G}$ by Proposition 3.2, we may assume that L is Zariski dense in $\mathbf{G}$. Set $G = \mathbf{G}(k)$ and $V = \mathbf{V}(k)$. By Theorem 3.23, there exists a normal k-subgroup $\mathbf{N} \lhd \mathbf{G}$ such that the image of L in $(\mathbf{G}/\mathbf{N})(k)$ is precompact and such that μ is supported on $\mathbf{V}^\mathbf{N} \cap V$. Since $\mathbf{N}$ acts trivially on $\mathbf{V}^\mathbf{N}$, $N = \mathbf{N}(k)$ acts trivially on $\mathbf{V}^\mathbf{N} \cap V$ and we have that $N \lhd L$ is a closed normal subgroup. It follows that the image of L in G/N is closed.

Therefore, the image of L in $(\mathbf{G}/\mathbf{N})(k)$ is closed whence compact and so L is almost algebraic in G. □

The second corollary is Borel's density theorem.

COROLLARY 3.26. *Let $\mathbf{G}$ be a connected algebraic k-group. Assume that for every proper normal k-subgroup $\mathbf{N} \lhd \mathbf{G}$, the group $\mathbf{G}(k)/\mathbf{N}(k)$ in noncompact.*

Then for any lattice $\Gamma < \mathbf{G}(k)$, we have that Γ is Zariski dense in $\mathbf{G}$.

PROOF. Set $G = \mathbf{G}(k)$. Denote by $\nu \in \mathrm{Prob}(G/\Gamma)$ the unique G-invariant Borel probability measure on G/Γ. Denote by $\mathbf{H}$ the Zariski closure of Γ in $\mathbf{G}$ and set $H = \mathbf{H}(k)$. Then $\mathbf{H} < \mathbf{G}$ is a k-subgroup by Proposition 3.2 and so $\mathbf{G}/\mathbf{H}$ is an algebraic k-$\mathbf{G}$-variety. Since $\Gamma < \mathbf{H}(k)$, we may consider the G-equivariant factor map $q : G/\Gamma \to G/H$. Regarding $G/H \hookrightarrow (\mathbf{G}/\mathbf{H})(k)$, we may view $\mu = q_*\nu \in \mathrm{Prob}((\mathbf{G}/\mathbf{H})(k))^G$ as a G-invariant Borel probability measure on $(\mathbf{G}/\mathbf{H})(k)$. Using the assumption, the k-normal subgroup $\mathbf{N} \lhd \mathbf{G}$ appearing in Theorem 3.23 is equal to $\mathbf{G}$ and we obtain that μ is supported on $(\mathbf{G}/\mathbf{H})^{\mathbf{G}} \cap (\mathbf{G}/\mathbf{H})(k)$. In particular, we have $(\mathbf{G}/\mathbf{H})^{\mathbf{G}} \neq \emptyset$ and so $\mathbf{H} = \mathbf{G}$. □

In particular, for any (almost) simple algebraic k-group $\mathbf{G}$ such that $\mathbf{G}(k)$ is noncompact, for any lattice $\Gamma < \mathbf{G}(k)$, we have that Γ is Zariski dense in $\mathbf{G}$.

The second main result of this subsection is Zimmer's tameness theorem (see [Zi84, Theorem 3.2.6]).

THEOREM 3.27 (Zimmer [Zi84]). *Let $\mathbf{G} < \mathrm{GL}_n$ be an algebraic k-group. Regard the projective space P^{n-1} as an algebraic k-$\mathbf{G}$-variety.*

Then the action $\mathbf{G}(k) \curvearrowright \mathrm{Prob}(\mathrm{P}^{n-1}(k))$ is tame.

Firstly, we show that the proof reduces to the case when $\mathbf{G} = \mathrm{GL}_n$.

CLAIM 3.28. Assume that the action $\mathrm{GL}_n \curvearrowright \mathrm{Prob}(\mathrm{P}^{n-1}(k))$ has locally closed orbits. Then for any algebraic k-subgroup $\mathbf{G} < \mathrm{GL}_n$, the action $\mathbf{G}(k) \curvearrowright \mathrm{Prob}(\mathrm{P}^{n-1}(k))$ is tame.

PROOF. Let $\mu \in \mathrm{Prob}(\mathrm{P}^{n-1}(k))$ be a Borel probability measure. Since the orbit $\mathrm{GL}_n(k)\mu$ is locally closed in $\mathrm{Prob}(\mathrm{P}^{n-1}(k))$, Proposition 2.12 implies that the map $\mathrm{GL}_n(k)/\mathrm{Stab}_{\mathrm{GL}_n(k)}(\mu) \to \mathrm{GL}_n(k)\mu : g\,\mathrm{Stab}_{\mathrm{GL}_n(k)}(\mu) \mapsto g\mu$ is a homeomorphism. Since $\mathbf{G}(k)\mu \subset \mathrm{GL}_n(k)\mu$, in order to show that $\mathbf{G}(k)\mu$ is locally closed in $\mathrm{Prob}(\mathrm{P}^{n-1}(k))$, it suffices to show that $\mathbf{G}(k)\mu$ is locally closed in $\mathrm{GL}_n(k)\mu$. Thus, using Theorem 2.13, it suffices to show that the action $\mathbf{G}(k) \curvearrowright \mathrm{GL}_n(k)/\mathrm{Stab}_{\mathrm{GL}_n(k)}(\mu)$ is tame. Moreover, using Proposition A.4, it suffices to show that action $\mathrm{Stab}_{\mathrm{GL}_n(k)}(\mu) \curvearrowright \mathrm{GL}_n(k)/\mathbf{G}(k)$ is tame.

Set $Z = \mathrm{GL}_n(k)/\mathbf{G}(k)$. By Corollary 3.25, there exists a k-subgroup $\mathbf{H} < \mathbf{G}$ such that $H = \mathbf{H}(k)$ sits as a cocompact closed normal subgroup in $L = \mathrm{Stab}_{\mathrm{GL}_n(k)}(\mu)$. By Theorem 3.21, the action $H \curvearrowright Z$ is tame. Since L/H

is a compact group and since $H\backslash Z$ is a standard Borel space, Proposition A.3 implies that the action $L/H \curvearrowright H\backslash Z$ is tame. It follows that the quotient Borel space $L\backslash Z \cong (L/H)\backslash(H\backslash Z)$ is countably separated. Therefore, the action $L \curvearrowright Z$ is tame. This finishes the proof of the claim. $\square$

Secondly, we consider the case when $\mathbf{G} = \mathrm{GL}_n$. We introduce some notation. The projective space $X = \mathrm{P}^{n-1}(k)$ is a compact metrizable space. Then $\mathrm{Prob}(X)$ endowed with the weak*-topology is a compact metrizable space. Denote by $\mathscr{C}$ the space of all closed subsets of X. Then $\mathscr{C}$ endowed with the Hausdorff metric is a compact metric space. For every $A \in \mathscr{C}$, we denote by $\mathrm{Prob}(A)$ the space of all Borel probability measures on X that are supported on A. Then $\mathrm{Prob}(A) \subset \mathrm{Prob}(X)$ is a closed subset. Whenever $\mathscr{A} \subset \mathscr{C}$ is a nonempty subset, we set $\mathrm{Prob}(\mathscr{A}) = \bigcup_{A \in \mathscr{A}} \mathrm{Prob}(A)$. We record the following easy lemma.

LEMMA 3.29. *Let $(A_j)_j$ be a sequence in $\mathscr{C}$ and $A \in \mathscr{C}$. For every $j \in \mathbb{N}$, let $\mu_j \in \mathrm{Prob}(A_j)$ and $\mu \in \mathrm{Prob}(X)$. Assume that $A_j \to A$ in $\mathscr{C}$ and that $\mu_j \to \mu$ in $\mathrm{Prob}(X)$. Then $\mu \in \mathrm{Prob}(A)$.*

In particular, if $\mathscr{A} \subset \mathscr{C}$ is closed, then $\mathrm{Prob}(\mathscr{A}) \subset \mathrm{Prob}(X)$ is closed.

PROOF. Let $f \in \mathrm{C}(X)$ be a continuous function such that $\mathrm{supp}(f) \cap A = \emptyset$. Since $A_j \to A$ in $\mathscr{C}$ with respect to the Hausdorff metric, there exists $j_0 \in \mathbb{N}$ such that $\mathrm{supp}(f) \cap A_j = \emptyset$ for every $j \geq j_0$. Then we have

$$\int_X f \, \mathrm{d}\mu = \lim_j \int_X f \, \mathrm{d}\mu_j = 0.$$

This shows that $\mu \in \mathrm{Prob}(A)$.

Next, assume that $\mathscr{A} \subset \mathscr{C}$ is closed. Let $(\mu_j)_{j \in \mathbb{N}}$ be a sequence in $\mathrm{Prob}(\mathscr{A})$ and $\mu \in \mathrm{Prob}(X)$ such that $\mu_j \to \mu$ in $\mathrm{Prob}(X)$. For every $j \in \mathbb{N}$, choose $A_j \in \mathscr{A}$ such that $\mu_j \in \mathrm{Prob}(A_j)$. Since $\mathscr{C}$ is a compact metric space and since $\mathscr{A} \subset \mathscr{C}$ is closed, upon taking a subsequence, we may assume that there exists $A \in \mathscr{A}$ such that $A_j \to A$ in $\mathscr{A}$. The previous result implies that $\mu = \lim_j \mu_j \in \mathrm{Prob}(A)$ and so $\mu \in \mathrm{Prob}(\mathscr{A})$. This shows that $\mathrm{Prob}(\mathscr{A}) \subset \mathrm{Prob}(X)$ is closed. $\square$

Consider the natural map $q : k^n \setminus \{0\} \to \mathrm{P}^{n-1}(k)$. For any nonzero subspace $V \subset k^n$, we denote by $[V] = q(V) \subset \mathrm{P}^{n-1}(k)$ the corresponding projective subspace. We record the following variation of a well-known result due to Furstenberg (see [**Fu62**, Lemma 1.5]).

Let $V \subset k^n$ be a nonzero subspace and set $r = \dim(V)$. Denote by $\mathrm{I}_{n,r}(k) \subset \mathrm{M}_{n,r}(k)$ the open subset of all injective linear maps $g : V \to k^n$. Consider the quotient space $\mathrm{I}_{n,r}(k)/k^*$ and denote by $\mathrm{I}_{n,r}(k) \to \mathrm{I}_{n,r}(k)/k^* : g \mapsto [g]$ the quotient map.

LEMMA 3.30. *Let $V \subset k^n$ be a nonzero subspace, $\mu \in \mathrm{Prob}([V])$ a Borel probability measure and $(g_j)_{j \in \mathbb{N}}$ a sequence in $\mathrm{I}_{n,r}(k)$. Assume that $[g_j V] \to [W]$ in $\mathscr{C}$ and that ${g_j}_*\mu \to \nu$ in $\mathrm{Prob}(\mathrm{P}^{n-1}(k))$.*

Then either the sequence $([g_j])_{j\in\mathbb{N}}$ *is precompact in* $\mathrm{I}_{n,r}(k)/k^*$ *or there exist nonzero subspaces* $Y, Z \subset W$ *such that* ν *is supported on* $[Y]\cup[Z]$ *with* $\dim(Y)+\dim(Z)=\dim(W)$.

Proof. Assume that $([g_j])_{j\in\mathbb{N}}$ is not precompact in $\mathrm{I}_{n,r}(k)/k^*$. For every $j\in\mathbb{N}$, set $h_j=\frac{g_j}{\|g_j\|}:V\to k^n$. Upon taking a subsequence, we may assume that there exists a linear map $h:V\to k^n$ such that $\lim_j\|h-h_j\|=0$ and $\ker(h)\neq\{0\}$. Set $\{0\}\neq N=\ker(h)\subset V$ and $\{0\}\neq Z=\mathrm{range}(h)\subset W$ so that we have $\dim(N)+\dim(Z)=\dim(V)=\dim(W)$. Upon taking a subsequence, we may assume that $[g_jN]\to[Y]$ where $Y\subset W$. We have $\dim(Y)+\dim(Z)=\dim(W)$. We claim that $\nu=\lim_j {g_j}_*\mu$ is supported on $[Y]\cup[Z]$. Write $\mu=\mu_1+\mu_2$ where $\mu_1=\mu|_{[N]}$ and $\mu_2=\mu|_{[V]\setminus[N]}$. Upon taking a subsequence, we may assume that for every $i\in\{1,2\}$, the limit $\nu_i=\lim_j {g_j}_*\mu_i$ exists so that we have $\nu=\nu_1+\nu_2$. It is clear that ν_1 is supported on $[Y]$. It remains to show that ν_2 is supported on $[Z]$. Let $f\in\mathrm{C}(\mathrm{P}^{n-1}(k))$ be a continuous function such that $\mathrm{supp}(f)\cap[Z]=\emptyset$. For every $j\in\mathbb{N}$, we may extend $g_j:k^n\to k^n$ to a linear map and we have

$$\begin{aligned}\int_{\mathrm{P}^{n-1}(k)} f\,\mathrm{d}\nu_2 &= \lim_j \int_{\mathrm{P}^{n-1}(k)} f\,\mathrm{d}{g_j}_*\mu_2\\ &= \lim_j \int_{\mathrm{P}^{n-1}(k)} f(g_jx)\,\mathrm{d}\mu_2(x)\\ &= \lim_j \int_{[V]\setminus[N]} f(g_jx)\,\mathrm{d}\mu_2(x).\end{aligned}$$

For every $x\in[V]\setminus[N]$, we have $\lim_j f(g_jx)=0$. Then Lebesgue's dominated convergence theorem implies that $\int_{\mathrm{P}^{n-1}(k)} f\,\mathrm{d}\nu_2=0$. Thus, ν_2 is supported on $[Z]$ and so $\nu=\nu_1+\nu_2$ is supported on $[Y]\cup[Z]$. □

Denote by $\mathscr{A}\subset\mathscr{C}$ the closed subset consisting of all elements of the form $A=\bigcup_{i=1}^{\ell}[V_i]$ where $V_i\subset k^n$ is a nonzero subspace such that $V_i\not\subset V_j$ for all $1\leq i\neq j\leq\ell$ and $\sum_{i=1}^{\ell}\dim(V_i)\leq n$. Set $\ell(A)=\ell$ and $d(A)=\sum_{i=1}^{\ell}\dim(V_i)$ and observe that $1\leq\ell(A),d(A)\leq n$.

We are now ready to prove Theorem 3.27.

Proof of Theorem 3.27. As we already explained, by Claim 3.28, we may assume that $\mathbf{G}=\mathrm{GL}_n$. By Proposition 2.12, It suffices to show that the action $\mathrm{GL}_n(k)\curvearrowright\mathrm{Prob}(\mathrm{P}^{n-1}(k))$ has locally closed orbits. Let $\mu\in\mathrm{Prob}(\mathrm{P}^{n-1}(k))$ be a Borel probability measure. Set

$$\begin{aligned}d(\mu)&=\min\{d(A)\mid A\in\mathscr{A}\text{ and }\mu\in\mathrm{Prob}(A)\}\\ \ell(\mu)&=\max\{\ell(A)\mid A\in\mathscr{A},\mu\in\mathrm{Prob}(A)\text{ and }d(A)=d(\mu)\}.\end{aligned}$$

Choose an element $A\in\mathscr{A}$ such that $\mu\in\mathrm{Prob}(A)$ and $d(A)=d(\mu)$ and $\ell(A)=\ell(\mu)$. Write $A=\bigcup_{i=1}^{\ell(\mu)}[V_i]$ so that $\sum_{i=1}^{\ell(\mu)}\dim(V_i)=d(\mu)$. Since $\mu\in\mathrm{Prob}([\sum_{i=1}^{\ell(\mu)}V_i])$ and by choice of $A\in\mathscr{A}$, it follows that $\dim(\sum_{i=1}^{\ell(\mu)}V_i)=$

$\sum_{i=1}^{\ell(\mu)} \dim(V_i) = d(\mu)$. In particular, the subspaces $V_1, \dots, V_{\ell(\mu)}$ are linearly independent.

Denote by $\mathscr{K}(\mu) \subset \mathscr{A}$ the subset of all elements of the form $B = \bigcup_{i=1}^{\ell(B)} [W_i] \in \mathscr{A}$ where $\ell(B) = \ell(\mu)+1$ and $d(B) = d(\mu)$, or $d(B) \leq d(\mu)-1$ or $\dim(\sum_{i=1}^{\ell(B)} W_i) \leq d(\mu)-1$. It is easy to see that $\mathscr{K}(\mu) \subset \mathrm{P}^{n-1}(k)$ is a closed subset. Set $\mathscr{U}(\mu) = \mathrm{Prob}(X) \setminus \mathrm{Prob}(\mathscr{K}(\mu))$, which is an open set by Lemma 3.29 and observe that $\mu \in \mathscr{U}(\mu)$. Moreover, $\mathscr{U}(\mu)$ is invariant under the natural action of $\mathrm{GL}_n(k)$. To show that the orbit $\mathrm{GL}_n(k)\mu \subset \mathrm{Prob}(\mathrm{P}^{n-1}(k))$ is locally closed, it suffices to show that $\mathrm{GL}_n(k)\mu = \overline{\mathrm{GL}_n(k)\mu} \cap \mathscr{U}(\mu)$. We clearly have $\mathrm{GL}_n(k)\mu \subset \overline{\mathrm{GL}_n(k)\mu} \cap \mathscr{U}(\mu)$. It remains to prove the inclusion $\overline{\mathrm{GL}_n(k)\mu} \cap \mathscr{U}(\mu) \subset \mathrm{GL}_n(k)\mu$.

Choose a sequence $(g_j)_{j \in \mathbb{N}}$ in $\mathrm{GL}_n(k)$ such that ${g_j}_*\mu \to \nu$ for some Borel probability measure $\nu \in \mathscr{U}(\mu)$. We show that $\nu \in \mathrm{GL}_n(k)\mu$. Upon taking a subsequence, we may further assume that $[g_j V_i] \to [W_i]$ in $\mathscr{C}$ for every $1 \leq i \leq \ell(\mu)$. Note that $\dim(W_i) = \dim(V_i)$ for every $1 \leq i \leq \ell(\mu)$. Moreover, since $\nu \in \mathrm{Prob}([\bigcup_{i=1}^{\ell(\mu)} W_i])$ and since $\nu \in \mathscr{U}(\mu)$, it follows that

$$d(\mu) \leq \dim(\sum_{i=1}^{\ell(\mu)} W_i) \leq \sum_{i=1}^{\ell(\mu)} \dim(W_i) = \sum_{i=1}^{\ell(\mu)} \dim(V_i) = d(\mu).$$

This further implies that the subspaces $W_1, \dots, W_{\ell(\mu)}$ are linearly independent. For every $1 \leq i \leq \ell(\mu)$, set $\mu_i = \mu|_{[V_i]}$, $\nu_i = \nu|_{[W_i]}$ and define the sequence $(h_i^j : V_i \to k^n)_{j \in \mathbb{N}}$ by the formula $h_i^j = g_j|_{V_i}$. Then for every $1 \leq i \leq \ell(\mu)$, we have $\lim_j {h_i^j}_* \mu_i = \nu_i$.

Claim 3.31. For every $1 \leq i \leq \ell(\mu)$, the sequence $([h_i^j])_{j \in \mathbb{N}}$ is precompact in $\mathrm{I}_{n,\dim(V_i)}(k)/k^*$.

Proof. By contradiction, assume that the sequence $([h_i^j])_{j \in \mathbb{N}}$ is not precompact in $\mathrm{I}_{n,\dim(V_i)}(k)/k^*$. Then Lemma 3.30 implies that there exist nonzero subspaces $Y_i, Z_i \subset W_i$ such that ν_i is supported on $[Y_i] \cup [Z_i]$ and $\dim(Y_i) + \dim(Z_i) = \dim(W_i) = \dim(V_i)$. There are two cases to consider:

- If $Y_i \cap Z_i = \{0\}$, then letting $B = [W_1] \cup \cdots \cup [W_{i-1}] \cup [Y_i] \cup [Z_i] \cup [W_{i+1}] \cup \cdots \cup [W_{\ell(\mu)}] \in \mathscr{A}$, we have $\nu \in \mathrm{Prob}(B)$, $\ell(B) = \ell(\mu) + 1$ and $d(B) = d(\mu)$. This contradicts the fact that $\nu \in \mathscr{U}(\mu)$.
- If $Y_i \cap Z_i \neq \{0\}$, then letting $B = [W_1] \cup \cdots \cup [W_{i-1}] \cup [Y_i + Z_i] \cup [W_{i+1}] \cup \cdots \cup [W_{\ell(\mu)}] \in \mathscr{A}$, we have $\nu \in \mathrm{Prob}(B)$ and $\dim(W_1) + \cdots + \dim(W_{i-1}) + \dim(Y_i + Z_i) + \dim(W_{i+1}) + \cdots + \dim(W_{\ell(\mu)}) \leq d(\mu) - 1$. This contradicts again the fact that $\nu \in \mathscr{U}(\mu)$.

This finishes the proof of the claim. $\square$

By Claim 3.31, upon taking a subsequence, for every $1 \leq i \leq \ell(\mu)$, we may assume that there exists a sequence $(\lambda_i^j)_{j \in \mathbb{N}}$ in k^* such that $\lambda_i^j h_i^j \to h_i$ as $j \to \infty$, where $h_i : V_i \to W_i$ is an isomorphism such that ${h_i}_* \mu_i = \nu_i$.

Choose any element $h \in \mathrm{GL}_n(k)$ such that $h|_{V_i} = h_i$ for every $1 \leq i \leq \ell(\mu)$. Then $\nu = h_*\mu \in \mathrm{GL}_n(k)\mu$. This finishes the proof of the theorem. □

CHAPTER 4

Margulis' superrigidity theorem

We prove Margulis' superrigidity theorem following the approach by Bader–Furman [**BF18a, BF18b**].

This chapter is devoted to proving the following superrigidity theorem for group homomorphisms due to Margulis (1975) (see [**Ma91**, Chapter 7]).

Margulis' superrigidity theorem. *Let* $\mathbf{H}$ *be a connected semisimple algebraic* $\mathbb{R}$*-group with* $\mathrm{rk}_{\mathbb{R}}(\mathbf{H}) \geq 2$*. Assume that* $H = \mathbf{H}(\mathbb{R})$ *has no compact factor. Let* $\Gamma < H$ *be an irreducible lattice.*

Let k *be a local field of characteristic zero,* $\mathbf{G}$ *a connected simple algebraic* k*-group and set* $G = \mathbf{G}(k)$*. Let* $\rho : \Gamma \to G$ *be a homomorphism such that* $\rho(\Gamma) < G$ *is Zariski dense and unbounded.*

Then there exists a unique continuous homomorphism $\overline{\rho} : H \to G$ *such that* $\overline{\rho}|_\Gamma = \rho$*.*

We will prove Margulis' superrigidity theorem in the case when $\mathbf{H}$ is also assumed to simple. More precisely, we will state and prove a superrigidity theorem due to Bader–Furman [**BF18a, BF18b**]. We will then derive Margulis' superrigidity theorem from Bader–Furman's superrigidity theorem.

We present Bader–Furman's approach to superrigidity that relies on the concept of *algebraic representation of ergodic actions.* For Margulis' proof of his superrigidity theorem, we refer the reader to [**Ma91**, Chapter 7] (see also [**Zi84**, Chapter 5] and [**Be08**, Chapter 10]).

1. Algebraic representations of ergodic actions

In this section, we follow the exposition given in [**BF18a**, Section 4] and [**BF18b**, Section 3]. Let T be a locally compact second countable group, (X, ν) a standard probability space and $T \curvearrowright (X, \nu)$ an ergodic action. Let k be a local field of characteristic zero and $\mathbf{G}$ an algebraic k-group. Let $\tau : T \to \mathbf{G}(k)$ be a continuous homomorphism. The following notion is central in Bader–Furman's approach (see [**BF18a**, Definition 4.1]).

Definition 4.1. An *algebraic representation* of $T \curvearrowright X$ is the data of an algebraic k-$\mathbf{G}$-variety $\mathbf{V}$ and a T-equivariant measurable map $\phi_{\mathbf{V}} : X \to \mathbf{V}(k)$.

The equivariance condition means that for every $t \in T$ and ν-almost every $x \in X$, we have $\phi_{\mathbf{V}}(tx) = \tau(t)\phi_{\mathbf{V}}(x)$.

We will simply refer to $\phi_{\mathbf{V}} : X \to \mathbf{V}(k)$ as the algebraic representation (of $T \curvearrowright X$). A *morphism* between $\phi_{\mathbf{U}}$ and $\phi_{\mathbf{V}}$ is the data of a $\mathbf{G}$-equivariant k-morphism $\pi : \mathbf{U} \to \mathbf{V}$ such that $\phi_{\mathbf{V}} = \pi \circ \phi_{\mathbf{U}}$ ν-almost everywhere. We say that $\phi_{\mathbf{V}}$ is a *coset* algebraic representation of $T \curvearrowright X$ if $\mathbf{V} = \mathbf{G}/\mathbf{H}$ where $\mathbf{H} < \mathbf{G}$ is a k-subgroup. The class of algebraic representations of $T \curvearrowright X$ and their morphisms forms a category. Firstly, we prove the existence of coset algebraic representations of $T \curvearrowright X$ (see [**BF18a**, Proposition 4.2]).

PROPOSITION 4.2. *Let $\phi_{\mathbf{V}}$ be an algebraic representation. Then there exist a k-subgroup $\mathbf{H} < \mathbf{G}$, a coset algebraic representation $\phi_{\mathbf{G}/\mathbf{H}}$ and a $\mathbf{G}$-equivariant k-morphism $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{V}$ such that $\phi_{\mathbf{V}} = \beta \circ \phi_{\mathbf{G}/\mathbf{H}}$ ν-almost everywhere.*

PROOF. Set $G = \mathbf{G}(k)$ and $V = \mathbf{V}(k)$. We denote by $p : V \to G\backslash V$ the quotient map. By Theorem 3.20, the Borel action $G \curvearrowright V$ is tame and the quotient Borel space $G\backslash V$ is standard. Since the map $p \circ \phi_{\mathbf{V}} : X \to G\backslash V$ is measurable and T-invariant and since the action $T \curvearrowright (X, \nu)$ is ergodic, $p \circ \phi_{\mathbf{V}}$ is ν-almost everywhere constant. Let $v \in V$ be a point such that $p \circ \phi_{\mathbf{V}} = Gv$ ν-almost everywhere. Thus, $\phi_{\mathbf{V}}(X)$ is essentially contained in Gv.

By Theorem 3.20, denote by $\mathbf{H} = \mathrm{Stab}_{\mathbf{G}}(v) < \mathbf{G}$ the stabilizer k-subgroup and set $H = \mathbf{H}(k)$. Regard $G/H \hookrightarrow (\mathbf{G}/\mathbf{H})(k)$. We obtain a $\mathbf{G}$-equivariant k-morphism $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{V}$ such that $\beta|_{G/H} : G/H \to Gv$ is a homeomorphism. Then the desired coset algebraic representation is $\phi_{\mathbf{G}/\mathbf{H}} = (\beta|_{G/H})^{-1} \circ \phi_{\mathbf{V}} : X \to G/H \hookrightarrow (\mathbf{G}/\mathbf{H})(k)$. $\square$

Secondly, we prove that the category of algebraic representations of $T \curvearrowright X$ has an initial object (see [**BF18a**, Theorem 4.3]).

THEOREM 4.3. *The category of algebraic representations of $T \curvearrowright X$ has an initial object that is a coset algebraic representation.*

PROOF. Consider the set $\mathscr{A}$ consisting of all algebraic subgroups $\mathbf{H} < \mathbf{G}$ for which there exists $g \in \mathbf{G}$ such that $g\mathbf{H}g^{-1} < \mathbf{G}$ is a k-subgroup and such that there exists a coset algebraic representation $\phi_{\mathbf{G}/g\mathbf{H}g^{-1}} : X \to (\mathbf{G}/g\mathbf{H}g^{-1})(k)$ of $T \curvearrowright X$. Note that $\mathbf{G} \in \mathscr{A}$ and so $\mathscr{A} \neq \emptyset$. Since the ring $K[\mathbf{G}]$ is Noetherian, $\mathscr{A}$ contains a minimal element $\mathbf{H}_{\min} < \mathbf{G}$. Choose $h \in \mathbf{G}$ such that $\mathbf{H}_0 = h\mathbf{H}_{\min}h^{-1} < \mathbf{G}$ is a k-subgroup and such that there exists a coset algebraic representation $\phi_0 : X \to (\mathbf{G}/\mathbf{H}_0)(k)$ of $T \curvearrowright X$. We show that the coset algebraic representation $\phi_0 : X \to (\mathbf{G}/\mathbf{H}_0)(k)$ is the required initial object.

Let $\phi_{\mathbf{V}} : X \to \mathbf{V}(k)$ be an algebraic representation of $T \curvearrowright X$. We need to show that there exists a unique $\mathbf{G}$-equivariant k-morphism $\beta : \mathbf{G}/\mathbf{H}_0 \to \mathbf{V}$ such that $\phi_{\mathbf{V}} = \beta \circ \phi_0$ ν-almost everywhere. By transitivity of $\mathbf{G} \curvearrowright \mathbf{G}/\mathbf{H}_0$, if such a $\mathbf{G}$-equivariant k-morphism $\beta : \mathbf{G}/\mathbf{H}_0 \to \mathbf{V}$ exists, it is necessarily unique. It remains to prove that such a $\mathbf{G}$-equivariant k-morphism $\beta : \mathbf{G}/\mathbf{H}_0 \to \mathbf{V}$ exists. To do this, we consider the product algebraic representation $\mathbf{W} = \mathbf{V} \times \mathbf{G}/\mathbf{H}_0$ with $\phi_{\mathbf{W}} = \phi_{\mathbf{V}} \times \phi_0$ ν-almost everywhere. By

Proposition 4.2, there exists a k-subgroup $\mathbf{H} < \mathbf{G}$, a coset algebraic representation $\phi_{\mathbf{G}/\mathbf{H}} : X \to (\mathbf{G}/\mathbf{H})(k)$ of $T \curvearrowright X$ and a $\mathbf{G}$-equivariant k-morphism $\theta : \mathbf{G}/\mathbf{H} \to \mathbf{W}$ such that $\phi_{\mathbf{W}} = \theta \circ \phi_{\mathbf{G}/\mathbf{H}}$ ν-almost everywhere. Consider the $\mathbf{G}$-equivariant k-morphism $p_2 \circ \theta : \mathbf{G}/\mathbf{H} \to \mathbf{G}/\mathbf{H}_0$. Let $g \in \mathbf{G}$ be such that $g\mathbf{H}_0 = (p_2 \circ \theta)(\mathbf{H}) \in (\mathbf{G}/\mathbf{H}_0)(k)$. Then $\mathbf{H} < g\mathbf{H}_0 g^{-1} = gh\mathbf{H}_{\min}(gh)^{-1}$. By minimality of $\mathbf{H}_{\min}$, it follows that $\mathbf{H} = gh\mathbf{H}_{\min}(gh)^{-1} = g\mathbf{H}_0 g^{-1}$ and so $p_2 \circ \theta$ is a k-isomorphism. Then $\beta = (p_1 \circ \theta) \circ (p_2 \circ \theta)^{-1} : \mathbf{G}/\mathbf{H}_0 \to \mathbf{V}$ is the required $\mathbf{G}$-equivariant k-morphism that satisfies $\phi_{\mathbf{V}} = \beta \circ \phi_0$ ν-almost everywhere. □

Following [**BF18a**], by a slight abuse of terminology, we call (a choice of) a coset algebraic representation that is an initial object in the category of algebraic representations of $T \curvearrowright X$ the *algebraic gate* of $T \curvearrowright X$ (even though the choice is not canonical in general).

Thirdly, we prove that the algebraic gate is nontrivial when the action $T \curvearrowright X$ is amenable and metrically ergodic (see [**BF18a**, Theorem 4.5] and [**BDL14**, Corollary 1.17]).

THEOREM 4.4. *Assume that* $\mathbf{G}$ *is a simple connected algebraic* k*-group,* $T \curvearrowright X$ *is amenable and metrically ergodic, and* $\tau(T) < \mathbf{G}(k)$ *is unbounded. Then there exists a coset algebraic representation* $\phi_{\mathbf{G}/\mathbf{H}}$ *of* $T \curvearrowright X$ *that is nontrivial in the sense that* $\mathbf{H} \neq \mathbf{G}$*.*

PROOF. Choose a faithful irreducible k-representation $\rho : \mathbf{G} \to \mathrm{GL}_n$. Note that since $\mathbf{G}$ is connected and simple, the adjoint k-representation $\mathrm{Ad} : \mathbf{G} \to \mathrm{GL}(\mathrm{Lie}(\mathbf{G}))$ is faithful and irreducible. Then we may regard $\mathbf{G} < \mathrm{GL}_n$ as a k-subgroup. Composing with the k-morphism $\mathrm{GL}_n \to \mathrm{PGL}_n$, we may further regard $\mathbf{G} < \mathrm{PGL}_n$ as a k-subgroup and $\mathbf{V} = \mathrm{P}^{n-1}$ as a k-$\mathbf{G}$-variety. Note that $(\mathrm{P}^{n-1})^{\mathbf{G}} = \emptyset$.

Set $G = \mathbf{G}(k)$. Since $V = \mathrm{P}^{n-1}(k)$ is compact and since $T \curvearrowright (X, \nu)$ is amenable, Theorem 2.42 implies that there exists a T-equivariant measurable map $\beta : X \to \mathrm{Prob}(V)$. By Theorem 3.27, the action $G \curvearrowright \mathrm{Prob}(V)$ is tame and the quotient Borel space $G\backslash \mathrm{Prob}(V)$ is standard. Denote by $p : \mathrm{Prob}(V) \to G\backslash \mathrm{Prob}(V)$ the quotient Borel map. Then the measurable map $p \circ \beta : X \to G\backslash \mathrm{Prob}(V)$ is T-invariant. Since $T \curvearrowright (X, \nu)$ is ergodic, $p \circ \beta : X \to G\backslash \mathrm{Prob}(V)$ is ν-almost everywhere constant and so there exists $\mu \in \mathrm{Prob}(V)$ such that $\beta(X)$ is essentially contained in $G\mu$. Set $L = \mathrm{Stab}_G(\mu) < G$. By Proposition 2.12, the orbit map $G/L \to G\mu$ is a homeomorphism and so we may regard $\beta : X \to G/L$ as a T-equivariant measurable map.

Denote by $\mathbf{H}$ the Zariski closure of L in $\mathbf{G}$. Proposition 3.2 implies that $\mathbf{H}$ is defined over k and we set $H = \mathbf{H}(k)$. By Theorem 3.23, there exists a k-subgroup $\mathbf{H}_0 < \mathbf{H} < \mathbf{G}$ such that $\mathbf{H}_0 \lhd \mathbf{H}$, the image of L is precompact in $(\mathbf{H}/\mathbf{H}_0)(k)$ and μ is supported on $(\mathrm{P}^{n-1})^{\mathbf{H}_0} \cap V$. Since $(\mathrm{P}^{n-1})^{\mathbf{G}} = \emptyset$, we have $\mathbf{H}_0 \neq \mathbf{G}$. We prove the following claim.

CLAIM 4.5. We have that $\mathbf{H}_0 \neq \{e\}$.

Proof of Claim 4.5. By contradiction, assume that $\mathbf{H}_0 = \{e\}$. Then $L < H$ is a compact subgroup. Choose a compatible proper left invariant metric d_G on G (see [**St73**]). Denote by $m_L \in \mathrm{Prob}(L)$ the Haar probability measure on L. Upon replacing d_G by the compatible right L-invariant metric

$$G \times G \to \mathbb{R}_+ : (g_1, g_2) \mapsto \int_L d_G(g_1\ell, g_2\ell)\, \mathrm{d}m_L(\ell),$$

we may assume that d_G is also right L-invariant. Set $Y = G/L$ and define

$$d_Y : Y \times Y \to \mathbb{R}_+ : (g_1L, g_2L) \mapsto \min_{(\ell_1,\ell_2)\in L\times L} d_G(g_1\ell_1, g_2\ell_2).$$

Then d_Y is a compatible G-invariant metric on Y and (Y, d_Y) is a separable metric space. Since $T \curvearrowright (X, \nu)$ is metrically ergodic, it follows that β is ν-almost everywhere constant and so there exists $g \in G$ such that $\beta = gL$ ν-almost everywhere. Since $\tau(T) < \mathrm{Stab}_G(gL) = gLg^{-1}$, this further implies that $\tau(T) < G$ is bounded, a contradiction. Therefore, we have $\mathbf{H}_0 \neq \{e\}$. □

Since $\mathbf{H}_0 \neq \mathbf{G}$ and $\mathbf{H}_0 \neq \{e\}$ by Claim 4.5, since $\mathbf{H}_0 \lhd \mathbf{H}$ and since $\mathbf{G}$ is simple, we have $\mathbf{H} \neq \mathbf{G}$. Since $L < H$, we may consider the G-equivariant factor map $q : G/L \to G/H$. Regarding $G/H \hookrightarrow (\mathbf{G}/\mathbf{H})(k)$, the map $\phi_{\mathbf{G}/\mathbf{H}} = q \circ \beta : X \to (\mathbf{G}/\mathbf{H})(k)$ is the desired nontrivial coset algebraic representation. □

Fourthly, we observe that in the case when the action $T \curvearrowright (X, \nu)$ is pmp and weakly mixing, the category of algebraic representations of $T \curvearrowright (X, \nu)$ is essentially trivial (see [**BF18b**, Proposition 3.3]).

Proposition 4.6. *Assume that the action $T \curvearrowright (X, \nu)$ is pmp and weakly mixing. Then any algebraic representation $\phi_{\mathbf{V}}$ of $T \curvearrowright X$ is ν-almost everywhere constant. Moreover, letting $\mathbf{H}$ the Zariski closure of $\tau(T)$ in $\mathbf{G}$, the essential image of $\phi_{\mathbf{V}}$ is $\mathbf{H}$-invariant.*

Proof. Denote by L the closure of $\tau(T)$ in G and by $\mathbf{H}$ the Zariski closure of L in $\mathbf{G}$. Set $\mu = {\phi_{\mathbf{V}}}_*\nu \in \mathrm{Prob}(\mathbf{V}(k))^L$. By Theorem 3.23, there exists a normal k-subgroup $\mathbf{N} \lhd \mathbf{H}$ such that the image of L in $(\mathbf{H}/\mathbf{N})(k)$ is compact and such that μ is supported on $\mathbf{V}^{\mathbf{N}} \cap \mathbf{V}(k)$. Denote by K the closure of the image of L in $(\mathbf{H}/\mathbf{N})(k)$. Then K is a compact group and the action $K \curvearrowright (\mathbf{V}^{\mathbf{N}} \cap \mathbf{V}(k), \mu)$ is well-defined, pmp and weakly mixing. Then a combination of Peter–Weyl theorem and Proposition 2.26 implies that $\mu = \delta_v$ for some point $v \in \mathbf{V}^{\mathbf{N}} \cap \mathbf{V}(k)$. By equivariance, it follows that $v \in \mathbf{V}^{\mathbf{N}} \cap \mathbf{V}(k)$ is $\mathbf{H}$-invariant. □

2. Algebraic representations of (S, T, Γ)

In this section, we follow the exposition given in [**BF18b**, Section 4]. In order to prove Bader–Furman's superrigidity theorem, we need to introduce a more sophisticated category of algebraic representations. Let S be a locally compact second countable group, $\Gamma < S$ a lattice and $T < S$ a closed subgroup. We endow S with its Haar measure m_S. Let k be a local field of characteristic zero and $\mathbf{G}$ an algebraic k-group. Set $G = \mathbf{G}(k)$ and fix a group homomorphism $\rho : \Gamma \to G$. The following notion is an extension of the notion of algebraic representation from Definition 4.1 (see [**BF18b**, Definition 4.1]).

DEFINITION 4.7. An *algebraic representation* of (S, T, Γ) consists of the following data:

- An algebraic k-group $\mathbf{L}$;
- A k-$(\mathbf{G} \times \mathbf{L})$-algebraic variety $\mathbf{V}$ where the $\mathbf{L}$-action is faithful. We regard $\mathbf{V}$ as a left-$\mathbf{G}$-right-$\mathbf{L}$-space.
- A continuous group homomorphism $\tau : T \to \mathbf{L}(k)$ with Zariski dense image.
- An algebraic representation $\phi_{\mathbf{V}} : S \to \mathbf{V}(k)$ of $\Gamma \times T \curvearrowright S$ (in the sense of Definition 4.1) where we regard S as a left-Γ-right-T-space. For every $\gamma \in \Gamma$, every $t \in T$ and m_S-almost every $s \in S$, we have

$$\phi_{\mathbf{V}}(\gamma s t) = \rho(\gamma)\phi_{\mathbf{V}}(s)\tau(t).$$

We simply refer to $\phi_{\mathbf{V}}$ as the algebraic representation of (S, T, Γ) denoting the extra data by $\mathbf{L}_{\mathbf{V}}$ and $\tau_{\mathbf{V}} : T \to \mathbf{L}_{\mathbf{V}}(k)$. A *morphism* between $\phi_{\mathbf{U}}$ and $\phi_{\mathbf{V}}$ is the data of a $(\mathbf{G} \times \mathbf{L}_{\mathbf{U},\mathbf{V}})$-equivariant k-morphism $\pi : \mathbf{U} \to \mathbf{V}$ such that $\phi_{\mathbf{V}} = \pi \circ \phi_{\mathbf{U}}$ ν-almost everywhere, where $\mathbf{L}_{\mathbf{U},\mathbf{V}} < \mathbf{L}_{\mathbf{U}} \times \mathbf{L}_{\mathbf{V}}$ is the Zariski closure of the image of $\tau_{\mathbf{U}} \times \tau_{\mathbf{V}} : T \to \mathbf{L}_{\mathbf{U}}(k) \times \mathbf{L}_{\mathbf{V}}(k)$. Note that $\mathbf{L}_{\mathbf{U},\mathbf{V}}$ naturally acts on $\mathbf{U}$ (resp. $\mathbf{V}$) via its projection to $\mathbf{L}_{\mathbf{U}}$ (resp. $\mathbf{L}_{\mathbf{V}}$).

Let $\mathbf{H} < \mathbf{G}$ be a k-subgroup and denote by $\mathbf{N} = \mathscr{N}_{\mathbf{G}}(\mathbf{H}) < \mathbf{G}$ the normalizer of $\mathbf{H}$ in $\mathbf{G}$, which is a k-subgroup by Proposition 3.15. Denote by $\mathrm{Aut}_{\mathbf{G}}(\mathbf{G}/\mathbf{H})$ the group of all $\mathbf{G}$-equivariant automorphisms of $\mathbf{G}/\mathbf{H}$. It is easy to see that the homomorphism

$$\mathbf{N} \to \mathrm{Aut}_{\mathbf{G}}(\mathbf{G}/\mathbf{H}) : n \mapsto \big(g\mathbf{H} \mapsto gn^{-1}\mathbf{H}\big)$$

is surjective and its kernel is equal to $\mathbf{H}$. Under the identification $\mathbf{N}/\mathbf{H} \cong \mathrm{Aut}_{\mathbf{G}}(\mathbf{G}/\mathbf{H})$, the group of k-points $(\mathbf{N}/\mathbf{H})(k)$ is identified with the group of k-$\mathbf{G}$-automorphisms of $\mathbf{G}/\mathbf{H}$.

We say that $\phi_{\mathbf{V}}$ is a *coset* algebraic representation of (S, T, Γ) if $\mathbf{V} = \mathbf{G}/\mathbf{H}$ where $\mathbf{H} < \mathbf{G}$ is a k-subgroup and $\mathbf{L} < \mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$ is a k-subgroup which acts on $\mathbf{G}/\mathbf{H}$ as described above. Firstly, we prove the existence of coset algebraic representations of (S, T, Γ). The next proposition should be compared with Proposition 4.2 (see [**BF18b**, Lemma 4.4]).

PROPOSITION 4.8. *Assume that the pmp action $T \curvearrowright S/\Gamma$ is weakly mixing. Let $\phi_{\mathbf{V}}$ be an algebraic representation of (S, T, Γ). Then there exists a k-subgroup $\mathbf{H} < \mathbf{G}$, a coset algebraic representation $\phi_{\mathbf{G}/\mathbf{H}}$ of (S, T, Γ) and an equivariant k-morphism $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{V}$ such that $\phi_{\mathbf{V}} = \beta \circ \phi_{\mathbf{G}/\mathbf{H}}$ almost everywhere.*

PROOF. Since $T \curvearrowright S/\Gamma$ is weakly mixing hence ergodic, it follows that $\Gamma \times T \curvearrowright S$ is ergodic. By Proposition 4.2, there exists a k-subgroup $\mathbf{M} <$

$\mathbf{G}\times\mathbf{L}$, a coset algebraic representation $\phi_{(\mathbf{G}\times\mathbf{L})/\mathbf{M}}$ of $\Gamma\times T \curvearrowright S$ and a $(\mathbf{G}\times\mathbf{L})$-equivariant k-morphism $\pi : (\mathbf{G}\times\mathbf{L})/\mathbf{M} \to \mathbf{V}$ such that $\phi_{\mathbf{V}} = \pi \circ \phi_{(\mathbf{G}\times\mathbf{L})/\mathbf{M}}$ almost everywhere. Thus, we may assume that $\mathbf{V} = (\mathbf{G}\times\mathbf{L})/\mathbf{M}$.

Denote by $p_2 : \mathbf{G}\times\mathbf{L} \to \mathbf{L}$ the projection map. Then $p_2(\mathbf{M}) < \mathbf{L}$ is a k-subgroup. By composing the $\mathbf{G}$-invariant-$\mathbf{L}$-equivariant k-morphism $\mathbf{V} \to \mathbf{L}/p_2(\mathbf{M})$ with the algebraic representation $\phi_{\mathbf{V}} : S \to \mathbf{V}(k)$, we obtain a well-defined algebraic representation $\phi : S/\Gamma \to (\mathbf{L}/p_2(\mathbf{M}))(k)$ of $T \curvearrowright S/\Gamma$. Since $T \curvearrowright S/\Gamma$ is pmp and weakly mixing and since $\tau_{\mathbf{V}}(T)$ is Zariski dense in $\mathbf{L}$, Proposition 4.6 implies that $p_2(\mathbf{M}) = \mathbf{L}$.

Set $\mathbf{H} = p_1(\mathbf{M} \cap \mathbf{G}\times\{e\})$. Then $\mathbf{H} < \mathbf{G}$ is a k-subgroup and the map

$$\beta : \mathbf{G}/\mathbf{H} \to \mathbf{V} : g\mathbf{H} \mapsto (g,e)\mathbf{M}$$

is a $\mathbf{G}$-equivariant k-isomorphism of k-$\mathbf{G}$-varieties. We may endow $\mathbf{G}/\mathbf{H}$ with a faithful $\mathbf{L}$-action by pulling back the faithful $\mathbf{L}$-action on $\mathbf{V}$ using $\beta : \mathbf{G}/\mathbf{H} \to \mathbf{V}$. Then $\phi_{\mathbf{G}/\mathbf{H}} = \beta^{-1} \circ \phi_{\mathbf{V}}$ is the desired coset algebraic representation of (S, T, Γ). □

Secondly, we prove that the category of algebraic representations of (S, T, Γ) has an initial object. The next theorem should be compared with Theorem 4.3 (see [**BF18b**, Theorem 4.3]).

THEOREM 4.9. *Assume that the pmp action $T \curvearrowright S/\Gamma$ is weakly mixing. Then the category of algebraic representations of (S, T, Γ) has an initial object that is a coset algebraic representation.*

PROOF. Consider the set $\mathscr{A}$ consisting of all algebraic subgroups $\mathbf{H} < \mathbf{G}$ for which there exists $g \in \mathbf{G}$ such that $g\mathbf{H}g^{-1} < \mathbf{G}$ is a k-subgroup and such that there exists a coset algebraic representation $\phi_{\mathbf{G}/g\mathbf{H}g^{-1}} : S \to (\mathbf{G}/g\mathbf{H}g^{-1})(k)$ of (S, T, Γ). Note that $\mathbf{G} \in \mathscr{A}$ and so $\mathscr{A} \neq \emptyset$. Since the ring $K[\mathbf{G}]$ is Noetherian, $\mathscr{A}$ contains a minimal element $\mathbf{H}_{\min} < \mathbf{G}$. Choose $h \in \mathbf{G}$ such that $\mathbf{H}_0 = h\mathbf{H}_{\min}h^{-1} < \mathbf{G}$ is a k-subgroup and such that there exists a coset algebraic representation $\phi_0 : S \to (\mathbf{G}/\mathbf{H}_0)(k)$ of (S, T, Γ). Denote by $\mathbf{L}_0 < \mathscr{N}_{\mathbf{G}}(\mathbf{H}_0)/\mathbf{H}_0$ the corresponding algebraic k-subgroup and by $\tau_0 : T \to \mathbf{L}_0(k)$ the corresponding homomorphism. We show that the coset algebraic representation $\phi_0 : S \to (\mathbf{G}/\mathbf{H}_0)(k)$ is the required initial object.

Let $\phi_{\mathbf{V}} : S \to \mathbf{V}(k)$ be an algebraic representation of (S, T, Γ). We need to show that there exists a unique equivariant k-morphism $\beta : \mathbf{G}/\mathbf{H}_0 \to \mathbf{V}$ such that $\phi_{\mathbf{V}} = \beta \circ \phi_0$ almost everywhere. By transitivity of $\mathbf{G} \curvearrowright \mathbf{G}/\mathbf{H}_0$, if such an equivariant k-morphism $\beta : \mathbf{G}/\mathbf{H}_0 \to \mathbf{V}$ exists, it is necessarily unique. It remains to prove that such an equivariant k-morphism $\beta : \mathbf{G}/\mathbf{H}_0 \to \mathbf{V}$ exists. To do this, we consider the product algebraic representation $\mathbf{W} = \mathbf{V} \times \mathbf{G}/\mathbf{H}_0$ with $\phi_{\mathbf{W}} = \phi_{\mathbf{V}} \times \phi_0$ almost everywhere, $\tau_{\mathbf{W}} = \tau_{\mathbf{V}} \times \tau_0$ and $\mathbf{L}_{\mathbf{W}}$ the Zariski closure of $\tau_{\mathbf{W}}(T)$ in $\mathbf{L}_{\mathbf{V}} \times \mathbf{L}_0$. By Proposition 4.8, there exists a k-subgroup $\mathbf{H} < \mathbf{G}$, a coset algebraic representation $\phi_{\mathbf{G}/\mathbf{H}} : S \to (\mathbf{G}/\mathbf{H})(k)$ of (S, T, Γ) and an equivariant k-morphism

$\theta : \mathbf{G}/\mathbf{H} \to \mathbf{W}$ such that $\phi_{\mathbf{W}} = \theta \circ \phi_{\mathbf{G}/\mathbf{H}}$ almost everywhere. Arguing as in the proof of Theorem 4.3, the $\mathbf{G}$-equivariant k-morphism $p_2 \circ \theta : \mathbf{G}/\mathbf{H} \to \mathbf{G}/\mathbf{H}_0$ is a k-isomorphism. Then $\beta = (p_1 \circ \theta) \circ (p_2 \circ \theta)^{-1} : \mathbf{G}/\mathbf{H}_0 \to \mathbf{V}$ is the required equivariant k-morphism that satisfies $\phi_{\mathbf{V}} = \beta \circ \phi_0$ almost everywhere. □

Thirdly, we prove that an initial object in the category of algebraic representations of (S, T, Γ) naturally extends to an algebraic representation of (S, N, Γ) where $N = \mathscr{N}_S(T)$ is the normalizer of T in S (see [**BF18b**, Theorem 4.6]).

THEOREM 4.10. *Assume that the pmp action $T \curvearrowright S/\Gamma$ is weakly mixing. Let $\phi = \phi_{\mathbf{G}/\mathbf{H}}$ be an initial object in the category of algebraic representations of (S, T, Γ) with k-subgroup $\mathbf{L} < \mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$ and continuous homomorphism $\tau : T \to \mathbf{L}(k)$. Set $N = \mathscr{N}_S(T)$.*

Then there exist a continuous homomorphism $\overline{\tau} : N \to (\mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H})(k)$ satisfying $\overline{\tau}|_T = \tau$ for which, letting $\overline{\mathbf{L}}$ be the Zariski closure of $\overline{\tau}(N)$ in $\mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$, the data

$$\phi_{\mathbf{G}/\mathbf{H}} : S \to (\mathbf{G}/\mathbf{H})(k), \quad \overline{\tau} : N \to \overline{\mathbf{L}}(k) < (\mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H})(k)$$

forms an algebraic representation of (S, N, Γ).

PROOF. Let $n \in N$. Consider $\tau_n : T \to \mathbf{L}(k) : t \mapsto ntn^{-1}$ and $\phi_n : S \to (\mathbf{G}/\mathbf{H})(k) : s \mapsto \phi(sn^{-1})$. For every $\gamma \in \Gamma$, every $t \in T$ and almost every $s \in S$, we have

$$\phi_n(\gamma st) = \phi(\gamma stn^{-1}) = \rho(\gamma)\phi(sn^{-1})\tau(ntn^{-1}) = \rho(\gamma)\phi_n(s)\tau_n(t).$$

It follows that $\phi_n : S \to (\mathbf{G}/\mathbf{H})(k)$ with $\tau_n : T \to \mathbf{L}(k)$ is an algebraic representation of (S, T, Γ). Since ϕ is an initial object in the category of algebraic representations of (S, T, Γ), there exists a unique equivariant k-morphism $\overline{\tau}(n) : \mathbf{G}/\mathbf{H} \to \mathbf{G}/\mathbf{H}$ such that $\overline{\tau}(n) \circ \phi = \phi_n$. We may regard $\overline{\tau}(n) \in (\mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H})(k)$ and we have $\phi(sn^{-1}) = \phi(s)\overline{\tau}(n)^{-1}$ for almost every $s \in S$. If $n = t \in T$, then we necessarily have $\overline{\tau}(t) = \tau(t)$. By uniqueness, we obtain a group homomorphism $\overline{\tau} : N \to (\mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H})(k)$ such that $\overline{\tau}|_T = \tau$.

CLAIM 4.11. $\overline{\tau} : N \to (\mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H})(k)$ is continuous.

PROOF OF CLAIM 4.11. We follow the proof of [**BF18a**, Theorem 4.7]. For simplicity, set $L = (\mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H})(k)$, $V = (\mathbf{G}/\mathbf{H})(k)$ and $\mathscr{V} = \mathrm{L}^0(S, V)$. Endowed with the topology of convergence in measure, $\mathscr{V}$ is a Polish space. Firstly, consider the action $L \curvearrowright \mathscr{V}$ defined by $(g \circ \psi)(s) = \psi(s)g^{-1}$ for every $g \in L$ and every $\psi \in \mathscr{V}$. Then the action $L \curvearrowright \mathscr{V}$ is free and continuous. By Theorem 3.21, the action $L \curvearrowright \mathscr{V}$ has locally closed orbits. Then for every $\psi \in \mathscr{V}$, the map $L \to L\psi : g \mapsto g \circ \psi$ is a homeomorphism. In particular, the map $\alpha : L\phi \to L : g\phi \mapsto g$ is continuous. Secondly, consider the action $N \curvearrowright \mathscr{V}$ defined by $(n\psi)(s) = \psi(sn)$ for every $n \in N$ and every $\psi \in \mathscr{V}$. Then the action $N \curvearrowright \mathscr{V}$ is continuous. Indeed, this follows from the fact that for every measurable subset $Y \subset S$, the map $N \to \mathbb{R}_+ : n \mapsto \nu(Yn \triangle Y)$ is continuous. In particular, the map $\beta : N \to \mathscr{V} : n \mapsto \phi_n$ is continuous.

By definition of $\overline{\tau} : N \to L$, for every $n \in N$, we have $\overline{\tau}(n) \circ \phi = \phi_n$ and so $\overline{\tau}(n) = (\alpha \circ \beta)(n)$. Thus, $\overline{\tau} : N \to L$ is continuous. □

Denote by $\overline{\mathbf{L}}$ the Zariski closure in $\mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$ of $\overline{\tau}(N)$. Observe that for every $\gamma \in \Gamma$, every $n \in N$ and almost every $s \in S$, we have

$$\phi(\gamma s n) = \phi_{n^{-1}}(\gamma s) = \phi(\gamma s)\overline{\tau}(n) = \rho(\gamma)\phi(s)\overline{\tau}(n).$$

Therefore, a combination of the above equation together with Claim 4.11 shows that $\phi : S \to (\mathbf{G}/\mathbf{H})(k)$ with $\overline{\tau} : N \to \overline{\mathbf{L}}(k)$ is an algebraic representation of (S, N, Γ). □

We infer the following useful consequence (see [**BF18b**, Corollary 4.7]).

Corollary 4.12. *For every $i \in \{1, 2\}$, let $T_i < S$ be a closed subgroup such that the pmp action $T_i \curvearrowright S/\Gamma$ is weakly mixing and denote by*

$$\phi_i : S \to (\mathbf{G}/\mathbf{H}_i)(k), \quad \tau_i : T_i \to \mathbf{L}_i(k) < (\mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H}_i)(k)$$

the initial object in the category of algebraic representations of (S, T_i, Γ).

If T_2 normalizes T_1, then there exists $g \in \mathbf{G}$ such that $g^{-1}\mathbf{H}_1 g < \mathbf{G}$ is defined over k and $\mathbf{H}_2 < g^{-1}\mathbf{H}_1 g$.

Proof. Assume that T_2 normalizes T_1, that is, $T_2 < \mathscr{N}_S(T_1)$. By Theorem 4.10, we may regard $\phi_1 : S \to (\mathbf{G}_1/\mathbf{H}_1)(k)$ as an algebraic representation of (S, T_2, Γ). Since $\phi_2 : S \to (\mathbf{G}_2/\mathbf{H}_2)(k)$ is an initial object in the category of algebraic representations of (S, T_2, Γ), there exists a $\mathbf{G}$-equivariant k-morphism $\pi : \mathbf{G}/\mathbf{H}_2 \to \mathbf{G}/\mathbf{H}_1$ such that $\phi_1 = \pi \circ \phi_2$ almost everywhere. This implies that there exists $g \in \mathbf{G}$ such that $g^{-1}\mathbf{H}_1 = \pi(\mathbf{H}_2) \in (\mathbf{G}/\mathbf{H}_1)(k)$. This implies that $g^{-1}\mathbf{H}_1 g < \mathbf{G}$ is defined over k and $\mathbf{H}_2 < g^{-1}\mathbf{H}_1 g$. □

3. Bader–Furman's superrigidity theorem

In order to state Bader–Furman's superrigidity theorem, we introduce the following adhoc terminology (see [**BF18b**]).

Definition 4.13. We say that a locally compact second countable group S satisfies the *higher rank condition* if there exist finitely many closed noncompact subgroups $T_0, \dots, T_n < S$ such that S is topologically generated by $\{T_0, \dots, T_n\}$, T_0 is amenable, and in a cyclic order, for every $i \in \{0, \dots, n\}$, T_{i+1} normalizes T_i inside S.

Example 4.14. Any connected simple real Lie group H with finite center and $\mathrm{rk}_{\mathbb{R}}(H) \geq 2$ satisfies the higher rank condition. For instance, when $H = \mathrm{SL}_3(\mathbb{R})$, we can use the following family of subgroups

$$\begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix}.$$

The main result of this section is Bader–Furman's superrigidity theorem (see [**BF18b**, Theorem 1.3]).

THEOREM 4.15 (Bader–Furman [**BF18b**]). *Let S be a locally compact second countable group that satisfies the dynamical dichotomy for isometric actions and the higher rank condition. Let $\Gamma < S$ be a lattice.*

Let k be a local field of characteristic zero, $\mathbf{G}$ a connected simple algebraic k-group and set $G = \mathbf{G}(k)$. Let $\rho : \Gamma \to G$ be a homomorphism such that $\rho(\Gamma) < G$ is Zariski dense and unbounded.

Then there exists a unique continuous homomorphism $\overline{\rho} : S \to G$ such that $\overline{\rho}|_\Gamma = \rho$.

In particular, for every $n \geq 3$, $S = \mathrm{SL}_n(\mathbb{R})$ satisfies the dynamical dichotomy for isometric actions by Theorem 2.5 and the higher rank condition by Example 4.14. Note that in case $\mathbf{H}$ is a simple algebraic $\mathbb{R}$-group with $\mathrm{rk}_{\mathbb{R}}(\mathbf{H}) \geq 2$, Theorem 4.15 applied to $S = \mathbf{H}(\mathbb{R})$ implies Margulis' superrigidity therorem.

The uniqueness part in Theorem 4.15 is a consequence of the following general result (see [**BF18a**, Lemma 6.1]).

LEMMA 4.16. *Let S be a locally compact second countable group and $\Gamma < S$ a lattice. Let k be a local field of characteristic zero and $\mathbf{G}$ a connected simple algebraic k-group. Let $\rho_1, \rho_2 : S \to \mathbf{G}(k)$ be continuous homomorphisms. Assume that $\rho_1(S) < \mathbf{G}(k)$ is Zariski dense and unbounded and that $\rho_1|_\Gamma = \rho_2|_\Gamma$. Then $\rho_1 = \rho_2$.*

PROOF. We start by proving the following claim.

CLAIM 4.17. We have that $\rho_1(\Gamma) = \rho_2(\Gamma)$ is Zariski dense in $\mathbf{G}$.

Indeed, denote by $L_1 < \mathbf{G}(k)$ the closure of $\rho_1(S)$ in $\mathbf{G}(k)$. By assumption, L_1 is Zariski dense in $\mathbf{G}$. By Theorem 3.23, there exists a normal k-subgroup $\mathbf{N}_1 \lhd \mathbf{G}$ such that the image of L_1 in $(\mathbf{G}/\mathbf{N}_1)(k)$ is precompact and such that for every algebraic k-$\mathbf{G}$-variety $\mathbf{V}$, we have $\mathrm{Prob}(\mathbf{V}(k))^{L_1} = \mathrm{Prob}(\mathbf{V}^{\mathbf{N}_1} \cap \mathbf{V}(k))$. Since $\mathbf{G}$ is simple, either $\mathbf{N}_1 = \{e\}$ or $\mathbf{N}_1 = \mathbf{G}$. Since $L_1 < \mathbf{G}(k)$ is unbounded, we necessarily have $\mathbf{N}_1 = \mathbf{G}$. Denote by $\mathbf{H}_1 < \mathbf{G}$ the Zariski closure of $\rho_1(\Gamma)$ in $\mathbf{G}$. Then $\mathbf{H}_1 < \mathbf{G}$ is a k-subgroup by Proposition 3.2 and $\mathbf{V}_1 = \mathbf{G}/\mathbf{H}_1$ is a k-$\mathbf{G}$-variety. The continuous homomorphism $\rho_1 : S \to \mathbf{G}(k)$ gives rise to an S-equivariant measurable map $S/\Gamma \to \mathbf{V}_1(k)$. Pushing forward the S-invariant Borel probability measure on S/Γ, we obtain an L_1-invariant Borel probability measure on $\mathbf{V}_1(k)$ which is necessarily supported on $\mathbf{V}_1^{\mathbf{G}} \cap \mathbf{V}_1(k)$. Then $\mathbf{V}_1^{\mathbf{G}} \neq \emptyset$ and so $\mathbf{H}_1 = \mathbf{G}$. This shows that $\rho_1(\Gamma) = \rho_2(\Gamma)$ is Zariski dense in $\mathbf{G}$.

Assume by contradiction that $\rho_1 \neq \rho_2$. Choose $s_0 \in S$ such that $\rho_1(s_0) \neq \rho_2(s_0)$. Consider the diagonal homomorphism $\rho = \rho_1 \times \rho_2 : S \to \mathbf{G}(k) \times$

$\mathbf{G}(k)$. Denote by L the closure of $\rho(S)$ in $\mathbf{G}(k) \times \mathbf{G}(k)$ and by $\mathbf{L}$ the Zariski closure of L in $\mathbf{G} \times \mathbf{G}$. Since $\rho_1|_\Gamma = \rho_2|_\Gamma$, Claim 4.17 implies that the Zariski closure of $\rho(\Gamma)$ in $\mathbf{G} \times \mathbf{G}$ is equal to the diagonal subgroup $\Delta_{\mathbf{G}} < \mathbf{G} \times \mathbf{G}$. In particular, we have $\Delta_{\mathbf{G}} < \mathbf{L}$. We show that $\mathbf{L} = \mathbf{G} \times \mathbf{G}$. Indeed, we have $(\rho_1(s_0), \rho_2(s_0)) \in \mathbf{L}(k)$ and $(\rho_2(s_0), \rho_2(s_0)) \in \mathbf{L}(k)$ and thus $(e, e) \neq (\rho_1(s_0)\rho_2(s_0)^{-1}, e) \in \mathbf{L} \cap (\mathbf{G} \times \{e\})$. Then $\mathbf{L} \cap (\mathbf{G} \times \{e\})$ is a nontrivial normal k-subgroup of $\mathbf{G} \times \{e\}$. Since $\mathbf{G}$ is simple, we conclude that $\mathbf{G} \times \{e\} < \mathbf{L}$. Likewise, we have $\{e\} \times \mathbf{G} < \mathbf{L}$ and so $\mathbf{L} = \mathbf{G} \times \mathbf{G}$.

Finally, we apply again Theorem 3.23 to the algebraic k-group $\mathbf{G} \times \mathbf{G}$ and the Zariski dense closed subgroup $L < \mathbf{G}(k) \times \mathbf{G}(k)$. Then there exists a normal k-subgroup $\mathbf{N} \lhd \mathbf{G} \times \mathbf{G}$ such that the image of L in $((\mathbf{G} \times \mathbf{G})/\mathbf{N})(k)$ is precompact and such that for every algebraic k-$(\mathbf{G} \times \mathbf{G})$-variety $\mathbf{V}$, we have $\mathrm{Prob}(\mathbf{V}(k))^L = \mathrm{Prob}(\mathbf{V}^{\mathbf{N}} \cap \mathbf{V}(k))$. Consider the k-$(\mathbf{G} \times \mathbf{G})$-variety $\mathbf{V} = (\mathbf{G} \times \mathbf{G})/\Delta_{\mathbf{G}}$. The continuous homomorphism $\rho : S \to \mathbf{G}(k) \times \mathbf{G}(k)$ gives rise to an S-equivariant measurable map $S/\Gamma \to \mathbf{V}(k)$. Pushing forward the S-invariant Borel probability measure on S/Γ, we obtain an L-invariant Borel probability measure on $\mathbf{V}(k)$ which is necessarily supported on $\mathbf{V}^{\mathbf{N}} \cap \mathbf{V}(k)$. Note that the nontrivial normal k-subgroups $\mathbf{G} \times \mathbf{G}$, $\mathbf{G} \times \{e\}$, $\{e\} \times \mathbf{G}$ have no fixed points on $\mathbf{V}$. Thus, we have $\mathbf{N} = \{e\}$ and so $L < \mathbf{G}(k) \times \mathbf{G}(k)$ is compact, which contradicts that $\rho_1(S)$ is unbounded in $\mathbf{G}(k)$. Therefore, we have $\rho_1 = \rho_2$. □

Before proving Theorem 4.15, we need the following technical result that allows us to assemble continuous homomorphisms $\tau_i : T_i \to G$ to obtain a continuous homomorphism $\tau : S \to G$ (see [**BF18b**, Lemma 5.1]).

Lemma 4.18. *Let S, G be locally compact second countable groups, (X, ν) a standard probability space and $S \curvearrowright (X, \nu)$ a nonsingular action. Let $(T_i)_{i \in \mathbb{N}}$ be a countable family of closed subgroups of S that topologically generate S.*

Let $\varphi : X \to G$ be a measurable map. For every $i \in \mathbb{N}$, let $\tau_i : T_i \to G$ be a continuous homomorphism. Assume that for every $i \in \mathbb{N}$, every $t \in T_i$ and ν-almost every $x \in X$, we have $\varphi(tx) = \varphi(x)\tau_i(t)^{-1}$.

Then there exists a continuous homomorphism $\tau : S \to G$ such that for every $i \in \mathbb{N}$, we have $\tau|_{T_i} = \tau_i$ and for every $s \in S$ and ν-almost every $x \in X$, we have $\varphi(sx) = \varphi(x)\tau(s)^{-1}$.

Proof. Consider the group $\mathscr{G} = \mathrm{L}^0(X, G)$ endowed with the topology of convergence in measure. Then $\mathscr{G}$ is a Polish group.

Firstly, consider the action $G \curvearrowright \mathscr{G}$ defined by $(g\psi)(x) = \psi(x)g^{-1}$ for every $g \in G$ and every $\psi \in \mathscr{G}$. Then the action $G \curvearrowright \mathscr{G}$ is free and continuous. Moreover, for every $\psi \in \mathscr{G}$, the G-orbit $G\psi \subset \mathscr{G}$ is closed and the map $G \to G\psi : g \mapsto g\psi$ is a homeomorphism. Indeed, let $(g_n)_n$ be a sequence in G, $\psi, \phi \in \mathscr{G}$ such that $g_n\psi \to \phi$ in $\mathscr{G}$. Up to extracting a subsequence, we may assume that $\psi(x)g_n^{-1} \to \phi(x)$ for ν-almost every $x \in X$. This implies that $g = \lim_n g_n$ exists in G and $\phi = g\psi$.

Secondly, consider the action $S \curvearrowright \mathscr{G}$ defined by $(s\psi)(x) = \psi(s^{-1}x)$ for every $s \in S$ and every $\psi \in \mathscr{G}$. Then the action $S \curvearrowright \mathscr{G}$ is continuous. Indeed, this follows from the fact that for every measurable subset $Y \subset S$, the map $S \to \mathbb{R}_+ : s \mapsto \nu(sY \triangle Y)$ is continuous.

By assumption, the T_i-orbit of $\varphi \in \mathscr{G}$ is contained in the G-orbit of $\varphi \in \mathscr{G}$. Since the G-orbit $G\varphi \subset \mathscr{G}$ is closed and since S is topologically generated by the family $(T_i)_{i\in\mathbb{N}}$, it follows that the S-orbit of $\varphi \in \mathscr{G}$ is contained in the G-orbit of $\varphi \in \mathscr{G}$. Thus, for every $s \in S$, there exists a unique $\tau(s) \in G$ such that $\varphi(sx) = \varphi(x)\tau(s)^{-1}$ for ν-almost every $x \in X$. Then the map $\tau : S \to G$ is a continuous group homomorphism such that $\tau|_{T_i} = \tau_i$ for every $i \in \mathbb{N}$. $\square$

We are now ready to prove Theorem 4.15.

Proof of Theorem 4.15. The uniqueness part follows from Lemma 4.16. It remains to prove the existence part. We identify $\{0, \dots, n\} = \mathbb{Z}/(n+1)\mathbb{Z}$.

Let $i \in \mathbb{Z}/(n+1)\mathbb{Z}$. Since S satisfies the dynamical dichotomy for isometric actions and since $T_i < S$ is a noncompact closed subgroup, Proposition 2.31 implies that the pmp action $T_i \curvearrowright S/\Gamma$ is weakly mixing. By Theorem 4.9, the category of algebraic representations of (S, T_i, Γ) has an initial object that is a coset algebraic representation and that we denote by

$$\phi_i : S \to (\mathbf{G}/\mathbf{H}_i)(k), \quad \tau_i : T_i \to \mathbf{L}_i(k) < (\mathscr{N}_{\mathbf{G}}(\mathbf{H}_i)/\mathbf{H}_i)(k).$$

Since S satisfies the dynamical dichotomy for isometric actions and since $T_0 < S$ is an amenable noncompact closed subgroup, a combination of Proposition 2.19 and Theorem 2.44 shows that the action $S \curvearrowright S/T_0$ is amenable and metrically ergodic. Since $\Gamma < S$ is a lattice, a combination of Propositions 2.20 and 2.45 shows that the action $\Gamma \curvearrowright S/T_0$ is amenable and metrically ergodic. By Theorems 4.3 and 4.4, the category of algebraic representations of $\Gamma \curvearrowright S/T_0$ has a nontrivial initial object that is a coset algebraic representation and that we denote by $\phi : S/T_0 \to (\mathbf{G}/\mathbf{H})(k)$ where $\mathbf{H} < \mathbf{G}$ is a proper k-subgroup. Letting $\mathbf{L} = \{e\} < \mathscr{N}_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$ be the trivial subgroup and $\tau : T_0 \to \mathbf{L}(k)$ be the trivial homomorphism, we may regard $\phi : S \to (\mathbf{G}/\mathbf{H})(k)$ as a coset algebraic representation of (S, T_0, Γ). Since $\mathbf{H} < \mathbf{G}$ is a proper k-subgroup and since $\phi_0 : S \to (\mathbf{G}/\mathbf{H}_0)(k)$ is an initial object in the category of algebraic representations of (S, T_0, Γ), it follows that $\mathbf{H}_0 < \mathbf{G}$ is a proper k-subgroup.

Applying Corollary 4.12, for every $i \in \mathbb{Z}/(n+1)\mathbb{Z}$, there exists $g_i \in \mathbf{G}$ for which the map $\pi_i : \mathbf{G}/\mathbf{H}_{i+1} \to \mathbf{G}/\mathbf{H}_i : g\mathbf{H}_{i+1} \mapsto gg_i^{-1}\mathbf{H}_i$ is a $\mathbf{G}$-equivariant k-morphism such that $\pi_i \circ \phi_{i+1} = \phi_i$ almost everywhere. In particular, for every $i \in \mathbb{Z}/(n+1)\mathbb{Z}$, we have that $g_i^{-1}\mathbf{H}_i g_i < \mathbf{G}$ is a k-subgroup and $\mathbf{H}_{i+1} < g_i^{-1}\mathbf{H}_i g_i$. This implies that $(g_1 \cdots g_{n+1})\mathbf{H}_0(g_1 \cdots g_{n+1})^{-1} < \mathbf{H}_0$ and by the descending chain condition, we obtain $(g_1 \cdots g_{n+1})\mathbf{H}_0(g_1 \cdots g_{n+1})^{-1} = \mathbf{H}_0$. This further implies that for every $i \in \mathbb{Z}/(n+1)\mathbb{Z}$, we have $\mathbf{H}_{i+1} = g_i^{-1}\mathbf{H}_i g_i$

and $\pi_i : \mathbf{G}/\mathbf{H}_{i+1} \to \mathbf{G}/\mathbf{H}_i : g\mathbf{H}_{i+1} \mapsto gg_i^{-1}\mathbf{H}_i$ is a $\mathbf{G}$-equivariant k-isomorphism. Therefore, for every $i \in \{0, \dots, n-1\}$, up to replacing ϕ_{i+1} by $\pi_i \circ \phi_{i+1} = \phi_i$, we may assume that all the k-subgroups $\mathbf{H}_i$ are equal to $\mathbf{H}_0$ and that all the equivariant measurable maps $\phi_i : S \to (\mathbf{G}/\mathbf{H}_i)(k)$ are equal to $\phi_0 : S \to (\mathbf{G}/\mathbf{H}_0)(k)$ almost everywhere.

For notational convenience, we simply write $\mathbf{H} = \mathbf{H}_0$ and $\phi = \phi_0 : S \to (\mathbf{G}/\mathbf{H})(k)$. Denote by $\mathbf{N} = \mathscr{N}_{\mathbf{G}}(\mathbf{H}) < \mathbf{G}$ the normalizer of $\mathbf{H}$ in $\mathbf{G}$, which is a k-subgroup by Proposition 3.15. Denote by $\mathbf{L} < \mathbf{N}/\mathbf{H}$ the algebraic subgroup generated by $\mathbf{L}_0, \dots, \mathbf{L}_n$. Then Proposition 3.1 implies that $\mathbf{L} < \mathbf{N}/\mathbf{H}$ is a k-subgroup. Denote by $\pi : \mathbf{N} \to \mathbf{N}/\mathbf{H}$ the quotient k-morphism and by $\widehat{\mathbf{L}} = \pi^{-1}(\mathbf{L}) < \mathbf{N}$, which is a k-subgroup. Then we have $\mathbf{H} < \widehat{\mathbf{L}} < \mathbf{N}$. The quotient k-morphism $q : \mathbf{G}/\mathbf{H} \to \mathbf{G}/\widehat{\mathbf{L}}$ is $\mathbf{L}_i$-invariant for every $i \in \{0, \dots, n\}$. This further implies that the measurable map $q \circ \phi : S \to (\mathbf{G}/\widehat{\mathbf{L}})(k)$ is T_i-invariant for every $i \in \{1, \dots, n\}$. Since S is topologically generated by $T_0, \dots, T_n$, it follows that $q \circ \phi : S \to (\mathbf{G}/\widehat{\mathbf{L}})(k)$ is S-invariant hence constant almost everywhere. The unique essential value of $q \circ \phi$ is then $\rho(\Gamma)$-invariant. Since $\rho(\Gamma)$ is Zariski dense in $\mathbf{G}$, it follows that the unique essential value of $q \circ \phi$ is $\mathbf{G}$-invariant. This implies that $\widehat{\mathbf{L}} = \mathbf{G}$. Since $\widehat{\mathbf{L}} < \mathbf{N} < \mathbf{G} = \widehat{\mathbf{L}}$, it follows that $\mathbf{N} = \mathbf{G}$. Then $\mathbf{H} \lhd \mathbf{G}$ is a normal k-subgroup. Since $\mathbf{G}$ is simple and since $\mathbf{H} \neq \mathbf{G}$, it follows that $\mathbf{H} = \{e\}$.

We have a measurable map $\phi : S \to \mathbf{G}(k)$ and continuous homomorphisms $\tau_i : T_i \to \mathbf{G}(k)$ for every $i \in \{0, \dots, n\}$ that satisfy

$$\phi(\gamma s t) = \rho(\gamma)\phi(s)\tau_i(t) \tag{4.1}$$

for every $\gamma \in \Gamma$, every $t \in T_i$ and almost every $s \in S$. Considering $\gamma = 1$ and applying Lemma 4.18, there exists a continuous homomorphism $\tau : S \to \mathbf{G}(k)$ such that $\tau|_{T_i} = \tau_i$ for every $i \in \{0, \dots, n\}$ and $\phi(st) = \phi(s)\tau(t)$ for every $t \in S$ and almost every $s \in S$. Since S is topologically generated by $T_0, \dots, T_n$, Equation (4.1) can be upgraded to

$$\phi(\gamma s t) = \rho(\gamma)\phi(s)\tau(t) \tag{4.2}$$

for every $\gamma \in \Gamma$, every $t \in S$ and almost every $s \in S$. Since $S \curvearrowright S$ is transitive, Lemma 2.17 implies that up to modifying $\phi : S \to \mathbf{G}(k)$ on a null set, we may assume that Equation (4.2) holds for every $\gamma \in \Gamma$, every $t \in S$ and every $s \in S$. Set $g = \phi(e) = \phi(s)\tau(s^{-1})$ for every $s \in S$. Then for every $s \in S$, we have $\phi(s) = g\tau(s)$. Applying Equation (4.2) to $\gamma \in \Gamma$ and $s = t = e$, we have

$$g\tau(\gamma) = \phi(\gamma) = \rho(\gamma)\phi(e) = \rho(\gamma)g.$$

Define the continuous homomorphism $\overline{\rho} : S \to \mathbf{G}(k)$ by the formula $\overline{\rho}(s) = g\tau(s)g^{-1}$ for every $s \in S$. Then $\overline{\rho}|_\Gamma = \rho$. This finishes the proof of the theorem. $\square$

CHAPTER 5

Applications

We apply the superrigidity theorem to prove Mostow–Margulis' rigidity theorem. We also state Margulis' arithmeticity theorem.

1. Mostow–Margulis' rigidity theorem

In this section, we use Margulis' superrigidity theorem to prove Mostow–Margulis' rigidity theorem for lattices in higher rank simple Lie groups

Mostow–Margulis' rigidity theorem. *For every $i \in \{1, 2\}$, let G_i be a connected simple real Lie group with trivial center and $\mathrm{rk}_{\mathbb{R}}(G_i) \geq 2$, and $\Gamma_i < G_i$ a lattice. Then any isomorphism $\rho : \Gamma_1 \to \Gamma_2$ extends to a Lie group isomorphism $\overline{\rho} : G_1 \to G_2$.*

Before proving the above theorem, we need the following well-known result showing that connected semisimple Lie groups with trivial center are quasi-algebraic (see e.g. [**Zi84**, Proposition 3]).

Proposition 5.1. *Let G be a connected semisimple (resp. simple) Lie group with trivial center. Then there exist $n \geq 1$ and a connected semisimple (resp. simple) $\mathbb{R}$-subgroup $\mathbf{G} < \mathrm{GL}_n$ with trivial center such that G and $\mathbf{G}(\mathbb{R})^0$ are isomorphic as Lie groups.*

Proof. Denote by $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ the complexified Lie algebra of G and consider the complexified adjoint representation $\mathrm{Ad}_{\mathbb{C}} : G \to \mathrm{GL}(\mathfrak{g})$. Then $\mathbf{G} = \mathrm{Aut}(\mathfrak{g})^0$ is a Zariski connected algebraic group defined over $\mathbb{R}$ and with trivial center. Moreover, we have $\mathrm{Ad}_{\mathbb{C}}(G) < \mathrm{Aut}(\mathfrak{g})^0 = \mathbf{G}$. If G is a semisimple (resp. simple) Lie group, then $\mathfrak{g}$ is a semisimple (resp. simple) complex Lie algebra and so $\mathbf{G}$ is a semisimple (resp. simple) algebraic group. Moreover, by Lie theory, we have $\mathrm{Ad}_{\mathbb{C}}(G) = \mathbf{G}(\mathbb{R})^0$. Therefore, G and $\mathbf{G}(\mathbb{R})^0$ are isomorphic as Lie groups. □

We are now ready to prove Mostow–Margulis' rigidity theorem.

Proof. Let $\rho : \Gamma_1 \to \Gamma_2$ be an isomorphism. For every $i \in \{1, 2\}$, using Proposition 5.1, we may choose a connected simple algebraic $\mathbb{R}$-group $\mathbf{G}_i$ such that $G_i = \mathbf{G}_i(\mathbb{R})^0$ as Lie groups. Since G_1 satisfies the dynamical dichotomy for isometric actions by Theorem 2.5 and the higher rank condition and since $\rho(\Gamma_1) = \Gamma_2 < \mathbf{G}_2(\mathbb{R})$ is Zariski dense by Corollary

3.26, Theorem 4.15 implies that there exists a unique continuous homomorphism $\overline{\rho} : G_1 \to \mathbf{G}_2(\mathbb{R})$ such that $\overline{\rho}|_{\Gamma_1} = \rho$. Since G_1 is connected, we necessarily have $\overline{\rho}(G_1) < \mathbf{G}_2(\mathbb{R})^0 = G_2$. Set $\theta = \rho^{-1} : \Gamma_2 \to \Gamma_1$. The same reasoning as above implies that there exists a unique continuous homomorphism $\overline{\theta} : G_2 \to G_1$ such that $\overline{\theta}|_{\Gamma_2} = \theta = \rho^{-1}$. Observe that $\varphi = \overline{\theta} \circ \overline{\rho} : G_1 \to G_1 < \mathbf{G}_1(\mathbb{R})$ is a continuous group homomorphism such that $\varphi(\Gamma_1) = \Gamma_1 < \mathbf{G}_1(\mathbb{R})$ is Zariski dense. Then Lemma 4.16 implies that $\overline{\theta} \circ \overline{\rho} = \varphi = \mathrm{id}_{G_1}$. Likewise, we have $\overline{\rho} \circ \overline{\theta} = \mathrm{id}_{G_2}$. This implies that $\overline{\rho} : G_1 \to G_2$ is a topological group isomorphism and so $\overline{\rho} : G_1 \to G_2$ is a Lie group isomorphism. □

2. Margulis' arithmeticity theorem

The following fundamental theorem is a particular case of a general result due to Borel–Harish-Chandra.

Theorem 5.2 (Borel–Harish-Chandra [**BHC61**]). *Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group. Then $\mathbf{G}(\mathbb{Z}) < \mathbf{G}(\mathbb{R})$ is a nonuniform lattice.*

One can then view Theorem 5.2 as a generalization of Theorem 1.19. We also mention that any noncompact connected semisimple Lie group contains both uniform and nonuniform lattices (see e.g. [**Ra72**, Chapter XIV]).

Let G be a locally compact second countable group and $H_1, H_2 < G$ closed subgroups. We say that H_1 and H_2 are *commensurable* if $H_1 \cap H_2$ has finite index in both H_1 and H_2. If $\Gamma_1, \Gamma_2 < G$ are commensurable discrete subgroups, then $\Gamma_1 < G$ is a lattice if and only if $\Gamma_2 < G$ is a lattice.

Lemma 5.3. *Let G be a locally compact second countable group, $\Gamma < G$ a lattice and $\varphi : G \to H$ a surjective continuous group homomophism with compact kernel. Then $\varphi(\Gamma) < H$ is a lattice.*

Proof. Firstly, we show that $\varphi(\Gamma) < H$ is discrete. Set $N = \ker(\varphi) \lhd G$ and denote by $\overline{\varphi} : G/N \to H : gN \mapsto \varphi(g)$ the corresponding continuous group isomorphism. Let $(\gamma_n)_n$ be a sequence in Γ such that $\varphi(\gamma_n) \to e$ in H. Then $\gamma_n N \to N$ in G/N and so there exists a sequence $(h_n)_n$ in N such that $\gamma_n h_n \to e$ in G. Upon extracting a subsequence, we may assume that $h_n \to h$ in N. Then $\gamma_n \to h^{-1}$ in G. Since $\Gamma < G$ is discrete, it follows that $\gamma_n = h^{-1} \in \Gamma \cap N$ eventually. This shows that $\varphi(\gamma_n) = e$ eventually. Thus, $\varphi(\Gamma) < H$ is discrete.

Secondly, consider the surjective continuous map $\Phi : G/\Gamma \to H/\varphi(\Gamma) = g\Gamma \mapsto \varphi(g\Gamma)$. Denote by $\nu \in \mathrm{Prob}(G/\Gamma)$ the unique G-invariant Borel probablility measure. Then $\Phi_*\nu \in \mathrm{Prob}(H/\varphi(\Gamma))$ is a H-invariant Borel probability measure. Therefore, $\varphi(\Gamma) < H$ is a lattice. □

We introduce the following important terminology.

Definition 5.4. Let H be a connected semisimple Lie group with trivial center and no compact factor and $\Gamma < H$ a lattice. We say that $\Gamma < H$ is *arithmetic* if there exists a Zariski connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ and a surjective continuous group homomorphism $\varphi : \mathbf{G}(\mathbb{R})^0 \to H$ with compact kernel such that $\varphi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0)$ and Γ are commensurable.

Finally, we state Margulis' celebrated arithmeticity theorem.

Margulis' arithmeticity theorem. *Let G be a connected semisimple Lie group with trivial center, no compact factor and such that* $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$. *Let $\Gamma < G$ be an irreducible lattice. Then $\Gamma < G$ is arithmetic.*

The proof relies on Margulis' superrigidity theorem. We refer the reader to [**Ma91**, Chapter 8], [**Zi84**, Chapter 6] or [**Be08**, Chapter 11] for further details.

APPENDIX A

Appendix

1. Tame actions

A *Borel space* Z is a space endowed with a σ-algebra $\mathscr{Z}$ of Borel subsets. A topological space X is naturally a Borel space endowed with the σ-algebra $\mathscr{X}$ generated by open sets. A Borel space Z is *countably separated* if there exists a countable family $(U_n)_{n\in\mathbb{N}}$ of Borel subsets of Z that separates the points in Z in the following sense: for all $z_1, z_2 \in Z$ such that $z_1 \neq z_2$, there exists $n \in \mathbb{N}$ such that $z_1 \in U_n$ and $z_2 \notin U_n$ or $z_1 \notin U_n$ and $z_2 \in U_n$. In that case, for every $z \in Z$, the singleton $\{z\} \subset Z$ is a Borel subset. A Borel space Z is *standard* if Z is Borel isomorphic to a Borel subset of a Polish space. A standard Borel space is either finite, countable or Borel isomorphic to the segment $[0, 1]$.

Let G be a locally compact second countable group, Z a standard Borel space and $G \curvearrowright Z$ a Borel action in the sense that the action map $G\times Z \to Z : (g, z) \mapsto gz$ is Borel. We denote by $G\backslash Z$ the quotient Borel space endowed with the quotient Borel structure and by $p : Z \to G\backslash Z$ the quotient Borel map. By Varadarajan's theorem (see e.g. [**Zi84**, Theorem 2.19]), there exist a compact metrizable space X, a continuous action $G \curvearrowright X$ and a G-equivariant injective Borel map $\iota : Z \to X$. This implies that for every $z \in Z$, the orbit $Gz \subset Z$ is a Borel subset and the stabilizer $\mathrm{Stab}_G(z) < G$ is a closed subgroup.

We say that the Borel action $G \curvearrowright Z$ is *tame* if the quotient Borel space $G\backslash Z$ is countably separated. Firstly, we recall the following useful result due to Kallman (see e.g. [**Zi84**, Theorem A.7]).

Theorem A.1. *Let G be a locally compact second countable group, Z a standard Borel space and $G \curvearrowright Z$ a tame Borel action. Then the quotient $G\backslash Z$ is a standard Borel space and there exists a Borel section $\iota : G\backslash Z \to Z$ for the Borel projection $p : Z \to G\backslash Z$.*

We derive the following useful consequence.

Corollary A.2. *Let G be a locally compact second countable group and $H < G$ a closed subgroup. Then there exists a Borel section $\iota : G/H \to G$ for the factor map $p : G \to G/H$.*

We record some useful properties of tame actions. Firstly, we observe that Borel actions of compact second countable groups are always tame.

Proposition A.3. *Let K be a compact second countable group. Then any Borel action $K \curvearrowright Z$ on a standard Borel space is tame.*

Proof. Let Z be a standard Borel space and $K \curvearrowright Z$ a Borel action. By Varadarajan's theorem (see e.g. [Zi84, Theorem 2.19]), there exist a compact metrizable space X, a continuous action $K \curvearrowright X$ and a K-equivariant injective Borel map $\iota : Z \to X$. By compactness and continuity, all K-orbits in X are compact hence locally closed. Proposition 2.1 implies that the action $K \curvearrowright X$ is tame, that is, $K\backslash X$ is countably separated. Considering the injective Borel map $\overline{\iota} : K\backslash Z \to K\backslash X$ and pulling back the countable separating family of Borel subsets in $K\backslash X$, it follows that $K\backslash Z$ is countably separated, that is, $K \curvearrowright Z$ is tame. □

Secondly, we investigate tameness for induced actions. Let G be a locally compact second countable group and $H < G$ a closed subgroup. By Corollary A.2, we may choose a Borel section $\iota : G/H \to G$ for the factor map $p : G \to G/H$ such that $\iota(H) = e$. Let Z be a standard Borel space and $H \curvearrowright Z$ a Borel action. Define the Borel map $\tau : G \times G/H \to H : (g, c) \mapsto \iota(gc)^{-1} g \iota(c)$ which satisfies the 1-cocycle relation:

$$\forall g_1, g_2 \in G, \forall c \in G/H, \quad \tau(g_1 g_2, c) = \tau(g_1, g_2 c)\tau(g_2, c).$$

Set $\mathrm{Ind}_H^G(Z) = G/H \times Z$ and define the *induced* Borel action $G \curvearrowright \mathrm{Ind}_H^G(Z)$ by the formula

$$\forall g \in G, \forall (c, z) \in \mathrm{Ind}_H^G(Z), \quad g \cdot (c, z) = (gc, \tau(g, c)z).$$

Assume that the Borel action $H \curvearrowright Z$ is the restriction of a Borel action $G \curvearrowright Z$. Consider the Borel space $G/H \times Z$ endowed with the diagonal Borel action $G \curvearrowright G/H \times Z$. Define the Borel isomorphism

$$\Theta : G/H \times Z \to \mathrm{Ind}_H^G(Z) : (c, z) \mapsto (c, \iota(c)^{-1} z).$$

Then it is easy to check that $\Theta : G/H \times Z \to \mathrm{Ind}_H^G(Z)$ is G-equivariant. In this case, we may and will identify the diagonal action $G \curvearrowright G/H \times Z$ with the induced action $G \curvearrowright \mathrm{Ind}_H^G(Z)$.

Proposition A.4. *Let G be a locally compact second countable group and $H < G$ a closed subgroup. Let Z be a standard Borel space and $H \curvearrowright Z$ a Borel action. Then $H \curvearrowright Z$ is tame if and only if $G \curvearrowright \mathrm{Ind}_H^G(Z)$ is tame.*

In particular, let $H_1, H_2 < G$ be closed subgroups. Then $H_1 \curvearrowright G/H_2$ is tame if and only if $H_2 \curvearrowright G/H_1$ is tame.

Proof. We prove the first assertion. Define the Borel map $\varphi : Z \to \mathrm{Ind}_H^G(Z) : z \mapsto (H, z)$. Since for every $h \in H$ and every $z \in Z$, we have

$$h \cdot \varphi(z) = h \cdot (H, z) = (hH, c(h, H)z) = (H, hz) = \varphi(hz),$$

it follows that $\overline{\varphi} : H\backslash Z \to G\backslash \mathrm{Ind}_H^G(Z) : Hz \mapsto G \cdot \varphi(z)$ is a well-defined Borel map. Next, define the Borel map $\psi : \mathrm{Ind}_H^G(Z) \to Z : (c, z) \mapsto z$. Since for every $g \in G$, every $c \in G/H$ and every $z \in Z$, we have

$$\psi(g \cdot (c, z)) = \psi(gc, \tau(g, c)z) = \tau(g, c)z = \tau(g, c)\psi(c, z),$$

it follows that $\overline{\psi} : G\backslash \mathrm{Ind}_H^G(Z) \to H\backslash Z : G \cdot (c, z) \mapsto H\psi(c, z)$ is a well-defined Borel map. It is straightforward to see that $\overline{\varphi}$ and $\overline{\psi}$ are inverse of one another. This further implies that $H \curvearrowright Z$ is tame if and only if $G \curvearrowright \mathrm{Ind}_H^G(Z)$ is tame.

Next, we prove the second assertion. Using the first assertion and since G/H_2 is a G-space, we have that $H_1 \curvearrowright G/H_2$ is tame if and only if $G \curvearrowright \mathrm{Ind}_{H_1}^G(G/H_2)$ is tame if and only if $G \curvearrowright G/H_1 \times G/H_2$ is tame. Likewise, we have that $H_2 \curvearrowright G/H_1$ is tame if and only if $G \curvearrowright G/H_2 \times G/H_1$ is tame. Therefore, $H_1 \curvearrowright G/H_2$ is tame if and only if $H_2 \curvearrowright G/H_1$ is tame □

2. Disintegration of measures

Recall that for any standard Borel space X, the space $\mathrm{Prob}(X)$ of Borel probability measures on X is again a standard Borel space.

THEOREM A.5 (Rohlin's disintegration theorem). *Let X, Y be standard Borel spaces and $p : X \to Y$ a Borel map. Let $\nu \in \mathrm{Prob}(X)$ be a Borel probability measure and set $\overline{\nu} = p_*\nu \in \mathrm{Prob}(Y)$. Then there exists a $\overline{\nu}$-essentially unique Borel map $Y \to \mathrm{Prob}(X) : y \mapsto \nu_y$ that satisfies the following two properties:*

(i) *For $\overline{\nu}$-almost every $y \in Y$, we have $\nu_y(p^{-1}(\{y\})) = 1$.*
(ii) *For every Borel subset $U \subset X$, we have*

$$\nu(U) = \int_Y \nu_y(U)\,\mathrm{d}\overline{\nu}(y).$$

The $\overline{\nu}$-essential uniqueness in Theorem A.5 means that for any Borel map $Y \to \mathrm{Prob}(X) : y \mapsto \eta_y$ that satisfies items (i) and (ii), we have $\nu_y = \eta_y$ for $\overline{\nu}$-almost every $y \in Y$. For item (ii), we usually simply write $\nu = \int_Y \nu_y\,\mathrm{d}\overline{\nu}(y)$.

COROLLARY A.6. *Let G be a locally compact second countable group, X a standard Borel space and $G \curvearrowright X$ a tame Borel action. Let $H < G$ be a closed subgroup and $\nu \in \mathrm{Prob}(X)$ an H-invariant Borel probability measure. Then there exists $x \in X$ and an H-invariant Borel probability measure $\eta \in \mathrm{Prob}(Gx)$.*

PROOF. Since $G \curvearrowright X$ is tame, the quotient space $Y = G\backslash X$ is a standard Borel space. Consider the Borel projection map $p : X \to Y$ and set $\overline{\nu} = p_*\nu$. By Theorem A.5, there exists a $\overline{\nu}$-essentially unique Borel map $Y \to \mathrm{Prob}(X) : y \mapsto \nu_y$ that satisfies items (i) and (ii) in Theorem A.5.

Let $h \in G$ and consider $h_*\nu \in \mathrm{Prob}(X)$. The Borel map $Y \to \mathrm{Prob}(X) : y \mapsto h_*\nu_y$ satisfies $(h_*\nu_y)(p^{-1}(\{y\})) = 1$ for $\overline{\nu}$-almost every $y \in Y$ and $h_*\nu = \int_Y h_*\nu_y\,\mathrm{d}\overline{\nu}(y)$. By essential uniqueness, it follows that $(h_*\nu)_y = h_*\nu_y$ for $\overline{\nu}$-almost every $y \in Y$. This implies that for every $h \in H$ and $\overline{\nu}$-almost every $y \in Y$, we have $h_*\nu_y = (h_*\nu)_y = \nu_y$. Since Y and $\mathrm{Prob}(X)$ are standard Borel spaces, Lemma 2.17 implies that there exists a $\overline{\nu}$-conull strictly G-invariant measurable subset $Y_0 \subset Y$ and a strictly G-equivariant measurable map $Y_0 \to \mathrm{Prob}(X) : y \mapsto \eta_y$ such that $\eta_y = \nu_y$ for $\overline{\nu}$-almost every $y \in Y_0$. Then we may choose $y = Gx \in Y_0$ such that $\eta = \eta_y = \nu_y \in \mathrm{Prob}(X)$ is H-invariant and satisfies $\eta(Gx) = 1$. $\square$

Bibliography

[BDL14] U. Bader, B. Duchesne, J. Lécureux, *Almost algebraic actions of algebraic groups and applications to algebraic representations.* Groups Geom. Dyn. **11** (2017), 705–738.

[BF14] U. Bader, A. Furman, *Boundaries, rigidity of representations, and Lyapunov exponents.* Proceedings of the International Congress of Mathematicians–Seoul 2014. Vol. III, 71–96, Kyung Moon Sa, Seoul, 2014.

[BF18a] U. Bader, A. Furman, *Super-rigidity and non-linearity for lattices in products.* Compos. Math. **156** (2020), 158–178.

[BF18b] U. Bader, A. Furman, *An extension of Margulis's superrigidity theorem.* Dynamics, geometry, number theory–the impact of Margulis on modern mathematics, 47–65, Univ. Chicago Press, Chicago, IL, 2022.

[BG14] U. Bader, T. Gelander, *Equicontinuous actions of semisimple groups.* Groups Geom. Dyn. **11** (2017), 1003–1039.

[Be08] Y. Benoist, *Réseaux des groupes de Lie.* Lecture notes: https://www.imo.universite-paris-saclay.fr/~benoist/prepubli/08m2p6ch1a13.pdf

[Be09] Y. Benoist, *Five lectures on lattices in semisimple Lie groups.* Géométries à courbure négative ou nulle, groupes discrets et rigidités, 117–176, Sémin. Congr., **18**, Soc. Math. France, Paris, 2009.

[Bo91] A. Borel, *Linear algebraic groups.* Second edition. Graduate Texts in Mathematics, **126**. Springer-Verlag, New York, 1991. xii+288 pp.

[BHC61] A. Borel, Harish-Chandra, *Arithmetic subgroups of algebraic groups.* Ann. of Math. **75** (1962), 485–535.

[Ca71] J.W.S. Cassels, *An introduction to the geometry of numbers.* Corrected reprint of the 1971 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1997. viii+344 pp.

[DE14] A. Deitmar, S. Echterhoff, *Principles of harmonic analysis.* Second edition. Universitext. Springer, Cham, 2014. xiv+332 pp.

[Fu62] H. Furstenberg, *A Poisson formula for semi-simple Lie groups.* Ann. of Math. **77** (1963), 335–386.

[HR79] E. Hewitt, K.A. Ross, *Abstract harmonic analysis. Vol.* I. Structure of topological groups, integration theory, group representations. Second edition. Grundlehren der Mathematischen Wissenschaften, **115**. Springer-Verlag, Berlin-New York, 1979. ix+519 pp.

[Hu75] J.E. Humphreys, *Linear algebraic groups.* Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975. xiv+247 pp.

[Ma91] G.A. Margulis, *Discrete subgroups of semisimple Lie groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **17**. Springer-Verlag, Berlin, 1991. x+388 pp.

[Ra72] M.S. Raghunathan, *Discrete subgroups of Lie groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **68**. Springer-Verlag, New York-Heidelberg, 1972. ix+227 pp.

[Ru87] W. Rudin, *Real and complex analysis.* Third edition. McGraw-Hill Book Co., New York, 1987. xiv+416 pp.

[Ru91] W. Rudin, *Functional analysis.* Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991. xviii+424 pp.

[Se65] J.-P. Serre, *Lie algebras and Lie groups.* 1964 lectures given at Harvard University. Second edition. Lecture Notes in Mathematics, **1500**. Springer-Verlag, Berlin, 1992. viii+168 pp.

[Sh97] Y. Shalom, *Invariant measures for algebraic actions, Zariski dense subgroups and Kazhdan's property* (T). Trans. Amer. Math. Soc. **351** (1999), 3387–3412.

[St73] R.A. Struble, *Metrics in locally compact groups.* Compositio Math. **28** (1974), 217–222.

[Ta03] M. Takesaki, *Theory of operator algebras.* III. Encyclopaedia of Mathematical Sciences, **127**. Operator Algebras and Non-commutative Geometry, 8. Springer-Verlag, Berlin, 2003. xxii+548 pp.

[Zi84] R.J. Zimmer, *Ergodic theory and semisimple groups.* Monographs in Mathematics, **81**. Birkhäuser Verlag, Basel, 1984. x+209 pp.